Section 1.3 The Derivative

Rate of Change

Definition: The **change** from A to B is B - A

Example 1: Finding Change

The revenue (in dollars) form the sale of *x* plastic planter boxes is given by $R(x) = 20x - 0.02x^2$, $0 \le x \le 1{,}000$. What is the change in revenue if production is changed from 100 planters to 300 planters? What is the average change in revenue for this production?

Definition: For a function f **,** then the **average rate of change of** f **from** $x = a$ **to** $x = a + h$ is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+h) - f(a)}{h}.$$

That means that the average rate of change equals the slope of the line passing through the two points.

Average and Instantaneous Velocity

Definition: The average velocity of any object is the change in distance divided by the change in time. Let f be the position function of some object, then the **average velocity** over the interval $(a, a+h)$ is given by

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{f(a+h)-f(a)}{h}$$

Example 2: Finding Velocity

Suppose a bowling ball is dropped from the top of a building that is 500 feet tall.

a) Find the average velocity of the bowling ball from $x = 2$ to $x = 3$ seconds.

b) Find the average velocity of the bowling ball over the interval from $x = 2$ to $x = 2 + h$ seconds.

c) Find the limit as $h \to 0$ of the expression you found in

d) Can you guess the velocity of the ball at exactly 3 seconds?

The **velocity** or instantaneous velocity is the limit of the above expression as $h \to 0$:

$$v(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$$

Slope of the Tangent Line

For a function f, the **average rate of change of** f **from** $x=a$ **to** $x=a+h$ is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+h) - f(a)}{h}.$$

That means that the average rate of change equals the slope of the **secant line** passing through the two points.

Example 3: Slope of a Secant Line

Let $f(x) = 2x^2$ to find the following:

a) Slope of the secant line for $a = 1$ and $h = 2$.
b) Slope of the secant line for $a = 1$ and $h = 1$.
c) Graph $f(x)$ and the two secant lines.
d) Find and simplify the slope of the secant line for $a = 1$ and h any nonzero real number.
e) Discuss possible interpretations of the limit in d).

Definition: The **tangent line** to the graph of the function $y = f(x)$ at the point $P(a, f(a))$ is the line that goes through P with slope

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This slope defines the slope of the graph at that point $P(a, f(a))$.

This means that the velocity of an object at time c, moving with position function f, is given by the slope of the tangent line to f at c.

The Derivative

Definition: The **derivative** f' **of the function** f **at** x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists.

A function f is **differentiable** at c if $f'(c)$ exists. A function f is **differentiable on an open interval** (a,b) if $f'(c)$ exists for every $c \in (a,b)$.

Facts:
1. The tangent line to $y = f(x)$ at $x = c$ goes through $(c, f(c))$ and has slope $f'(c)$.
2. The derivative $f'(c)$ is the instantaneous rate of change of y with respect to x.

Other notations for the derivative include the following:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

Procedure: Four Step Process to Find the Derivative
1. Find $f(x+h)$.
2. Find $f(x+h) - f(x)$.
3. Find $\frac{f(x+h) - f(x)}{h}$.
4. Find $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Example 4: Finding a Derivative

Use the definition to find the derivative of the following functions.

a) $f(x) = x^2 - x$ Then use the result to find the slope of the tangent line to the graph for $x = -2, 0, 3$. Then find the equation of the tangent line to the graph at *x* = 3.

b) $f(x) = \dfrac{1}{x}$

c) $f(x) = \sqrt{x}$

Example 5: Application

The total sales of a company (in millions of dollars) t months from now are given by $S(t) = \sqrt{x} + 2$. Find $S(25)$ and $S'(25)$, and interpret. Use these results to predict sales after 26 months, 27 months.

Theorem: Differentiability and Continuity. If a function f is differentiable at a point $x = c$, then f is continuous at c.

There are three ways a function f can fail to be differentiable at a point $x = c$:

1. The graph of f is discontinuous at $x = c$.
2. The graph of f has a vertical tangent line at $x = c$.
3. The graph of f has a "corner" at $x = c$. This means the left and right-hand limits are not the same, so the slope of the curve on the left of $x = c$ is different from the slope of the curve to the right of $x = c$.

Example 6: Determining Differentiability

Find the slope of $f(x) = |x|$ on each side of $x = 0$. Use the definition to find the derivative of f at $x = 0$. Is f differentiable for $x = 0$? Is f differentiable for other values of x?