Section 2: Continuity

Definitions: A function f is **continuous at a number c** if

$$\lim_{x \to c} f(x) = f(c)$$

If f is not continuous at c, we say that the function is **discontinuous** at c or that the function has a **discontinuity** at c.

Example 1: Showing a Function is Discontinuous

Show that the function $f(x) = \begin{cases} \dfrac{x^2 - 4}{x + 2} & x \neq -2 \\ 0 & x = -2 \end{cases}$ is discontinuous at $x = -2$.

Solution

To see this, we find $\lim_{x \to -2} f(x)$.

Types of Discontinuities: There are three categories of discontinuities: removable discontinuities, jump discontinuities, and infinite discontinuities.

1. **Removable discontinuities** occur when there is a hole in the graph. In this case, the limit exists but the function is either not defined or is defined to be a number different than the limiting value.

   Example: $f(x) = \begin{cases} \dfrac{x^2 - 4}{x + 2} & x \neq -2 \\ 0 & x = -2 \end{cases}$

2. **Jump discontinuities** occur when the function jumps from one y value to another at a given point. In this case, the limit does not exist because the left and right-hand limits approach different values.

   Example: $f(x) = \frac{|x|}{x}$ has a jump discontinuity at $x = 0$.

3. **Infinite discontinuities** correspond to vertical asymptotes. In this case, the limit does not exist because one or both of the left or right-hand limits grows without bound.

   Example: $f(x) = \frac{x-2}{x^2-4}$ has a removable discontinuity at $x = 2$ and an infinite discontinuity at $x = -2$.

Definitions: A function f is **continuous on an interval** (a,b) **if** it is continuous for all $x \in (a,b)$.

We can also define continuity on an interval, which includes one or both endpoints by using the idea of continuity from the left and right-hand sides. For example, a function f is continuous on an interval $[a,b)$ **if** it is continuous for all $x \in (a,b)$ and continuous from the right at a.

Theorem: If f and g are continuous at a point c and k is a constant, then the following functions are continuous at c also:

1. $f+g$
2. $f-g$
3. kf
4. fg
5. $\frac{f}{g}$, If $g(c) \neq 0$

Theorem: Continuity of Polynomial and Rational Functions

(a) Any polynomial is continuous for all $x \in \mathbb{R}$.

(b) Any rational function is continuous for all real numbers x in its domain.

(c) Any root function is continuous for all real numbers x in its domain.

(d) Any exponential function is continuous for all x in its domain.

(e) Any logarithmic function is continuous for all x in its domain.

Example 3: Functions and Continuity

Where are the following functions continuous? For each of the functions that are discontinuous, discuss the type of discontinuity.

a) $f(x) = x^2 - 3x + 5$

b) $g(x) = \dfrac{3x + 5}{x^2 - 4}$

c) $f(x) = \sqrt[3]{x^2 - 9}$

d) $F(x) = \sqrt{x - 3}$

Solving Rational Inequalities Using Continuity

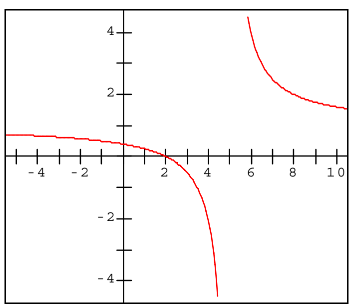
Definition: A **rational inequality** is an inequality of the form $R > 0$ where *R* is a rational expression.

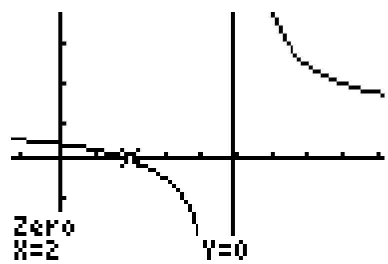
To solve a rational inequality graphically, draw the graph, find the x-intercepts, and then determine where the graph is positive and negative.

Example 4: Solving a rational inequality graphically

Solve the inequality $\dfrac{x-2}{x-5} < 0$ graphically.

Solution: The graph of $f(x) = \dfrac{x-2}{x-5}$ is shown below.





The only x-intercept is at $x = 2$. Notice that the function changes sign only at the x-intercept and at the vertical asymptote. Since the graph is above the x-axis everywhere except between 2 and 5. The solution is $(2,5)$.

Theorem: A rational function can only change signs at an x-intercept and at a point of discontinuity.

To solve a rational inequality symbolically, we do the following:

1. Put the inequality in standard form, i.e. $R > 0$.
2. Find the zeros (x-intercepts) and vertical asymptotes of the rational function. These are called the **critical (or partition) numbers**.
3. Plot the critical numbers on a number line. These numbers determine test intervals.
4. Choose a test number in each test interval and determine whether the inequality is true or false on that interval.
5. Write the solution using interval notation.

Example 5: Solving a rational inequality graphically

Solve the inequality $\frac{x^2 - 1}{x - 4} < 0$ symbolically.