Chapter 1 Limits

Section 1: The Limit of a Function

In this section we will consider what happens to the output values of a function as the inputs get close to a certain number.

Example 1: Examining a Limit Numerically and Graphically

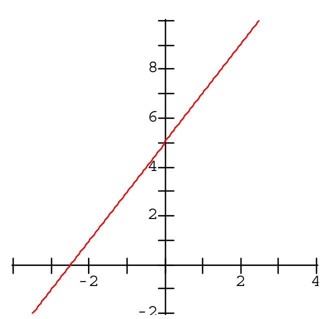
Examine the behavior of the function $f(x) = 2x + 5$ for values of x close to, but not equal to, 2. Do this both numerically and graphically.

Solution

Numerically: Fill in the following table to determine what happens to the values of the function as x gets close to 2.

x	$f(x)$
1.9	
1.9999	
1.9999999	
2	N/A
2.0000001	
2.00001	
2.001	
2.1	

Graphically: Use the graph to determine what happens to the values of $f(x) = 2x + 5$ for x values close to $x = 2$.



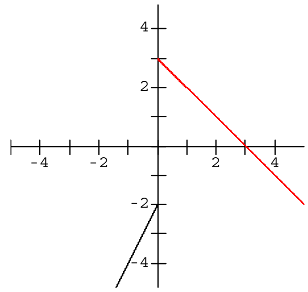
Definition: We write $\lim_{x \to c} f(x) = L$ and say that the limit of $f(x)$ as x approaches c is equal to L. This means we can make the values of $f(x)$ as close to L as we would like by choosing x close to but not equal to c.

Example 2: Limits

Try to find $\lim_{x \to 0} \frac{|x|}{x}$ graphically.

Example 3: Finding One-Sided Limits from a Graph

Examine the behavior of the function graphed below for values of x close to, but not equal to, 0.



Solution

As x approaches 0 from the left side, the values of $f(x)$ get close to -2. As x approaches 0 from the right side, the values of $f(x)$ get close to 3. These are examples of one-sided limits.

Definition: **One–sided limits.** We write $\lim_{x\to c^-} f(x) = L$ and say that the left-hand limit of $f(x)$ as x approaches c is equal to L. This means we can make the values of $f(x)$ as close to L as we would like by choosing x close to but less than c. If x approaches c from above (or from the right), we write $\lim_{x\to c^+} f(x) = L$ and say that the right-hand limit of $f(x)$ as x approaches c is equal to L. This means we can make the values of $f(x)$ as close to L as we would like by choosing x close to but not equal to c.

Example 4: One-sided limits

For the function in Example 4, find

a) $\lim_{x\to 0^-} f(x)$

b) $\lim_{x\to 0^+} f(x)$

Solution

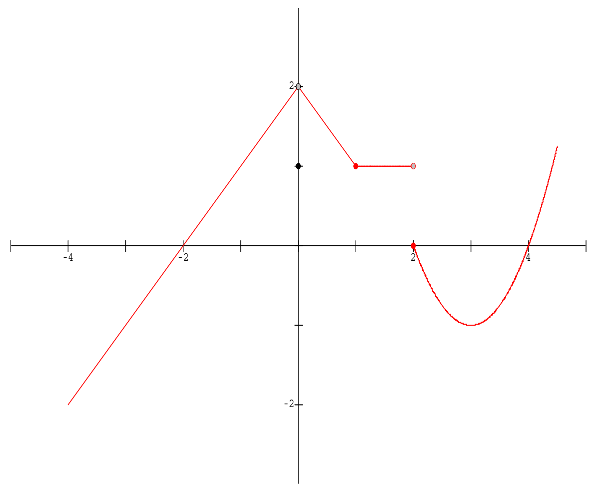
The left-hand limit at $x = 0$ is $\lim_{x\to 0^-} f(x) = -2$. The right-hand limit at $x = 0$ is $\lim_{x\to 0^+} f(x) = 3$. Since the function values approach two different values as we approach 0 from the left and from the right, we will say that the function does not have a limit at $x = 0$.

Theorem: $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = L$

Example 5: Evaluating Limits Graphically

Find the indicated limits, if they exist. If they do not exist, explain why they do not.

a) $\lim_{x \to -2} f(x)$



b) $\lim_{x \to 0} f(x)$

c) $\lim_{x \to 1} f(x)$

d) $\lim_{x \to 2^-} f(x)$

e) $\lim_{x \to 2^+} f(x)$

f) $\lim_{x \to 2} f(x)$

g) $\lim_{x \to 3} f(x)$

Example 6: Evaluating One-sided Limits

Examine the following limits by looking at the left-hand and right-hand limits.

a) $\lim_{x \to 0} \frac{1}{x}$

b) $\lim_{x \to 0} \frac{1}{x^2}$

Example 7: Calculating Limits

Calculate the following limits for each of the given functions:

i. $\lim_{x \to 2^-} f(x)$ ii. $\lim_{x \to 2^+} f(x)$ iii. $\lim_{x \to 2} f(x)$

a) $f(x) = \dfrac{|x-2|}{x-2}$

b) $f(x) = \dfrac{x^2-4}{x-2}$

c) $f(x) = \dfrac{x-2}{x^2-4}$

d) $f(x) = \dfrac{1}{(x-2)^2}$

Example 9: Calculating Limits Algebraically

Find the following limits.

a) $\lim\limits_{x \to -2} \dfrac{2x+4}{x^2-4}$

b) $\lim\limits_{t \to 0} \dfrac{\sqrt{4+t}-2}{t}$

Theorem: Properties of Limits

Suppose that k is a constant and the limits

$\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ both exist. Then the following are true:

1. $\lim_{x \to c}[f(x)+g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$ Example: $\lim_{x \to 0}(\sqrt{x^2+4}+x)$

2. $\lim_{x \to c}[f(x)-g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$ Example: $\lim_{x \to 0}(\sqrt{x^2+4}+x)$

3. $\lim_{x \to c}[kf(x)] = k \lim_{x \to c} f(x)$ Example: $\lim_{x \to 4}(2x)$

4. $\lim_{x \to c}[f(x)g(x)] = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$ Example: $\lim_{x \to 5}(x^2+2)(x-5)$

5. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ if $\lim_{x \to c} g(x) \neq 0$ Example: $\lim_{x \to 2} \dfrac{x+2}{x+4}$

6. $\lim_{x \to c}[f(x)]^n = [\lim_{x \to c} f(x)]^n$ Example: $\lim_{x \to \pi}(\ln x)^5$

7. $\lim_{x \to c} k = k$ Example: $\lim_{x \to 5} 7$

8. $\lim_{x \to c} x = c$ Example: $\lim_{x \to 5} x$

9. $\lim_{x \to c} x^n = c^n$ for any positive integer n Example: $\lim_{x \to 2} x^3$

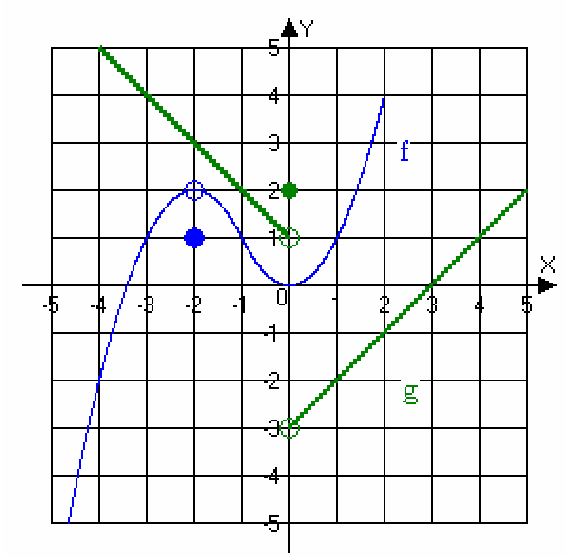
10. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$ for any positive integer n except when n is even we exclude negative values of c. Example: $\lim_{x \to 4} \sqrt{x}$

11. $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$ for any positive integer n except when n is even we assume that $\lim_{x \to c} f(x) \geq 0$.

    Example: $\lim_{x \to 8} \sqrt[4]{2x} = \sqrt[4]{\lim_{x \to 8} 2x}$

Example 10: Using the Properties of Limits

Use the graphs of the function f and g given below and the properties of limits to find the indicated limits.



a) $\lim_{x\to-2}[f(x)-3g(x)]$

b) $\lim_{x\to0}[f(x)g(x)]$

c) $\lim_{x\to1}\left[\dfrac{f(x)}{g(x)}\right]$

d) $\lim_{x\to3}\left[\dfrac{f(x)}{g(x)}\right]$

Example 11: Calculating Limits Using the Properties of Limits

Find $\lim_{x \to 3} \left(x^2 - x - 6 \right)$ carefully justifying each step.

Example 12: Calculating Limits Using the Properties of Limits

Find $\lim_{x \to 3} \dfrac{x^2 - x - 6}{x - 3}$

Example 13: Calculating Limits Using the Properties of Limits

Find $\lim_{x \to 3} \dfrac{x^2 + x}{x^2 - 3x}$

Example 14: Finding Limits of Piecewise-defined Functions

Find the indicated limits.

a) Find $\lim\limits_{x\to 2^+} f(x), \lim\limits_{x\to 2^-} f(x), \lim\limits_{x\to 2} f(x)$ if $f(x) = \begin{cases} 2x-3 & x \le 2 \\ -x^3+4x+1 & x > 2 \end{cases}$

Limits of Difference Quotients

Example 15: Calculating Limits Using the Properties of Limits

Find $\lim\limits_{h\to 0} \dfrac{(4+h)^2 - 16}{h}$