Huff

Name ____________________ Test 3 Business Calculus 1 Mike Huff Fall 2011

Show all work on the test paper for partial credit.

remember:

$(fg)' = fg' + gf'$

$(\ln x)' = \frac{1}{x}$

$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

(9 points)

1. Find the <u>second derivative</u> of the following functions:

a) $f(x) = x \ln x$

$f'(x) = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x$

$f''(x) = \frac{1}{x} = x^{-1}$

$f'''(x) = -x^{-2} = -\frac{1}{x^2}$

b) $f(x) = \dfrac{\ln x}{x}$

$f'(x) = \dfrac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$

$f''(x) = \dfrac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)2x}{(x^2)^2} = \dfrac{-3x + 2x\ln x}{x^4}$

c) $f(x) = \dfrac{e^x}{x+1}$

$f'(x) = \dfrac{(x+1)e^x - e^x}{(x+1)^2} = \dfrac{xe^x}{(x+1)^2}$

note: $(xe^x)' = xe^x + e^x$

$f''(x) = \dfrac{(x+1)^2(xe^x + e^x) - xe^x(2(x+1))}{(x+1)^4}$

The Derivative and Graphs (10 points)

8. Let $f(x) = x^3 - 4x^2 + 5x - 2$.

a) Find the domain of f.

$\mathbb{R}$ - all real #'s (the domain of all polynomials are $\mathbb{R}$)

b) Find $f'(x)$.

$f'(x) = 3x^2 - 8x + 5$

c) For what intervals is the graph of f increasing? Decreasing?

$3x^2 - 8x + 5 = 0$ → inc $(-\infty, 1) \cup (5/3, \infty)$

$(3x-5)(x-1)$ dec. $(1, 5/3)$

$3x - 5 = 0$ or $x - 1 = 0$

$x = \frac{5}{3}$ $x = 1$

inc dec. inc.

$+++++0------0++++$

1 5/3

4/3

d) Find any local maxima and local minima.

local max at $x = 1$ is $f(1) = 0$

local min at $x = 5/3$ is $f(5/3) = -\frac{4}{27}$

e) Find $f''(x)$.

$f''(x) = 6x - 8$

f) For what intervals is the graph of f concave upward? Concave downward?

$6x - 8 = 0$

$6x = 8 \Rightarrow x = \frac{8}{6}$

$x = 4/3$

f'' concave down $------0++++++$ concave up

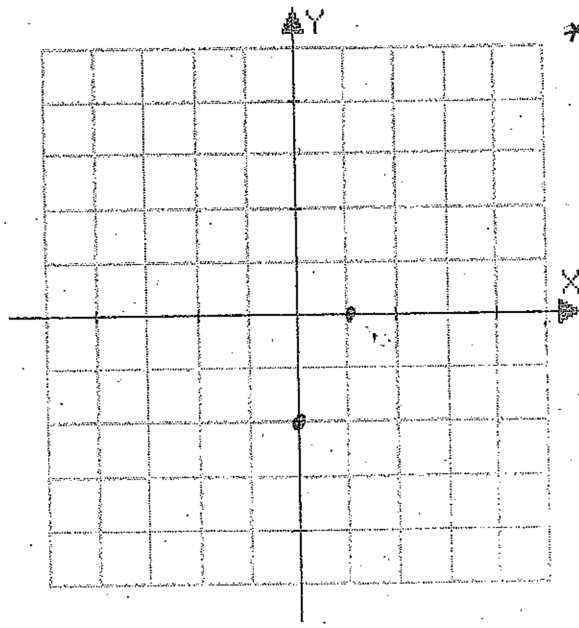
4/3

g) Find any inflections points on the graph of f.

inflection point $(4/3, -2/27)$

note: all cubic polynomials have one inflection point

a) Sketch the graph of the function f.



H.A:

$$\lim_{x\to\infty} \frac{2x/x^2}{\frac{x^2}{x^2}-\frac{9}{x^2}} = \lim_{x\to\infty} \frac{\cancel{2/x}^{\,0}}{1-\cancel{\frac{9}{x^2}}_{\,0}} = \frac{0}{1} = 0$$

$y=0$ is a H.A.

Curve Sketching Techniques (10 points)

9. Let $f(x) = \dfrac{2x}{x^2-9}$. ← Rational function

a) Find $f'(x)$.

$$f'(x) = \frac{(x^2-9)2-2x(2x)}{(x^2-9)^2} = \frac{2x^2-18-4x^2}{(x^2-9)^2} = \frac{-2x^2-18}{(x^2-9)^2} = \frac{-2(x^2+9)}{(x^2-9)^2} < 0$$

b) Find $f''(x)$.

$$f''(x) = \frac{(x^2-9)^2(-4x)+2(x^2+9)2(x^2-9)2x}{(x^2-9)^4}$$

(simplify)

c) Find the domain of f.

$x \neq \pm 3$ or $\{x \in \mathbb{R} \mid x \neq \pm 3\}$

d) Sketch the graph of the function f on the axes below.

b) For what intervals is the graph f increasing? Decreasing?

dec $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$
inc N/A
(always decreasing)

c) For what intervals is the graph of f concave upward? Concave downward? Is there an inflection point? If so, where?

d) Find the x-intercept.

$f(x)=0$
$2x=0$
$x=0 \to (0,0)$

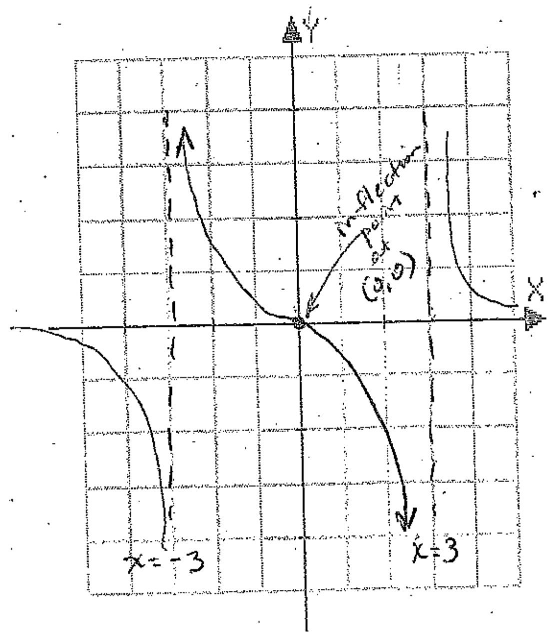
e) Find the y-intercept.

$f(0)=0$
$(0,0)$

f) Find the horizontal asymptote. → $y=0$

g) The vertical asymptote(s) is/are $x=-3;\ x=3$

h) Sketch the graph of the function f.



9.(cont.)

$$\frac{-4x(x^2-9)+8x(x^2+9)}{(x^2-9)^3} = \frac{-4x^3+36x+8x^3+72x}{(x^2-9)^3}$$

$$= \frac{4x^3+108x}{(x^2-9)^3} = \frac{4x(x^2+27)}{(x^2-9)^3}$$

c)

f'': concave down $(-)$, und at -3, concave up $(+)$, 0 at 0, concave down $(-)$, und at 3, concave up $(+)$

$-3 \qquad 0 \qquad 3$

Concave up $(-3,0)\cup(3,\infty)$
Concave down $(-\infty,-3)\cup(0,3)$

(9 points)

2. Let $f(x) = \dfrac{x}{\ln x}$, and answer the following:

note! domain $(0,1)\cup(1,\infty)$

$f(1)$ = undefined

a) Find $f'(x)$.

$$f'(x) = \frac{\ln x - x\cdot\frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$$

✓

b) On what interval(s) is the function increasing? Decreasing?

local min.

$0 = \ln x - 1$

$\ln x = 1$

$x = e^1$

*plug in values in original function to determine where it is inc/dec. (sign chart)

f': dec. ------ und. dec. ------ 0 inc +++++++

0, $1/2$, 1, e, e^2

$\ln e^2 - 1 = 2 - 1 > 0$

c) Find $f''(x)$.

$$f''(x) = \frac{(\ln x)^2\frac{1}{x} - (\ln x - 1)\,2\,\cancel{(\ln x)}\,\frac{1}{x}}{(\ln x)^{\cancel{4}\,3}}$$

Simplify)

$$\frac{(\ln x)\frac{1}{x} - \frac{2}{x}(\ln x - 1)}{(\ln x)^3}\cdot\frac{x}{x}$$

$$= \frac{\ln x - 2\ln x + 2}{x(\ln x)^3} = \frac{2 - \ln x}{x(\ln x)^3} = 0$$

⇓

$2 - \ln x = 0$

$\ln x = 2$

$x = e^2$

(sign chart)

f'': ------ und. +++++ 0 ------

0, 1, e^2

(e) (e^3)

d) On what interval(s) is the function concave up? Concave down?

f'': ------ und. +++++ 0 ------ concave down

0, 1, e^2

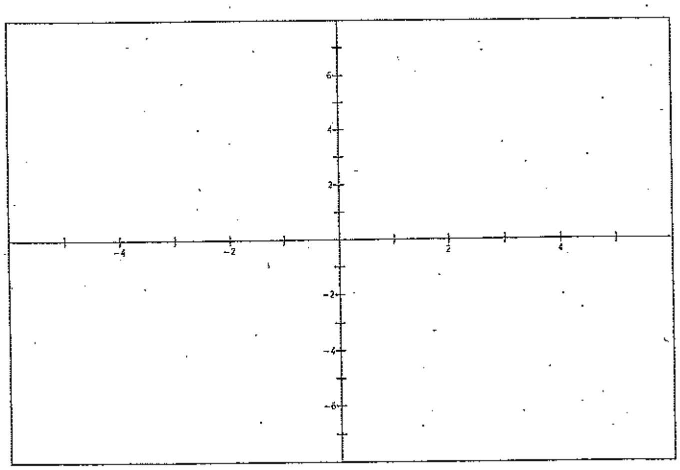
(e) (e^3)

*L.H's rule $\frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{x}{\ln x}$$

e) Find the absolute minimum of the function.

*



(6 points)

3. The cost of producing x units of a product is given by the cost function $C(x) = 200 + 50x - 50\ln x\,, x \geq 1$ Find the **minimum average cost.**

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{200 + 50x - 50\ln x}{x}$$

$$= \frac{200}{x} + 50 - \frac{50\ln x}{x}$$

$$= 200x^{-1} + 50 - \frac{50\ln x}{x}$$

$$\bar{C}'(x) = -200x^{-2} - \frac{x\frac{50}{x} - 50\ln x}{x^2}$$

$$= \frac{-200}{x^2} - \frac{50 - 50\ln x}{x^2}$$

$$= \frac{-250 + 50\ln x}{x^2} = 0$$

$$50\ln x - 250 = 0.$$

$$\ln x = \frac{250}{50} \qquad \ln x = 5$$

$$x = e^5$$

min at $x = e^5$

f' dec. $- - - -$ 0 $+ + + +$ inc.

e^3 e^5 e^6

(8 points)

4. Find the second derivative $\dfrac{d^2y}{dx^2}$ of the following functions.

a) $y = \ln\left(x^2 + x\right)$

$$y' = \frac{2x+1}{x^2+x}$$

$$y'' = \frac{(x^2+x)2 - (2x+1)(2x+1)}{(x^2+x)^2}$$

b) $y = e^{x^2}$

$$y' = 2xe^{x^2}$$

$$y'' = 2x\left[2xe^{x^2}\right] + e^{x^2}(2)$$

(6 points)

5. Find $f'(x)$ for the following functions

a) $f(x) = \log_5 x$.

b) $f(x) = 5^x$.

(7 points)

6. Suppose the price-demand equation for x units of a product is determined to be $p = 1000e^{-0.02x}$ where x units are sold per day at a price of $\$p$ each.

a) Find the production level that maximizes revenue.

$p = 1000e^{-.02x}$

$R(x) = xp = 1000xe^{-.02x}$

$R'(x) = 1000x(-.02e^{-.02x}) + 1000e^{-0.2x}$

$= (-20x + 1000)\cancel{e^{-.02x}} = 0$

$20x = 1000$

$\boxed{x = 50}$

$R'(x)$ +++++++0------

0 50

max rev. at $x = 50$ is $R(50)$

$p = \$367.88$

* b) Find the price for the product that maximizes revenue.

$R(50) = \boxed{\$367.88}$

(9 points)

7. Find the indicated limits.

LH's $0/0$

a) $\lim_{x\to 1} \frac{x^2-1}{x^4-1} = \lim_{x\to 1} \frac{2x}{4x^3} = \lim_{x\to 1} \frac{1}{2x^2} = \frac{1}{2}$

b) $\lim_{x\to 1} \frac{\ln x}{x^2-1}$

L.H's $\frac{0}{0}$

$= \lim_{x\to 1} \frac{\frac{1}{x}}{2x} = \lim_{x\to 1} \frac{1}{2x^2} = \frac{1}{2}$

c) $\lim_{x\to\infty} \frac{e^{3x}}{x^2}$

$= \lim_{x\to\infty} \frac{3e^{3x}}{2x} = \lim_{x\to\infty} \frac{9e^{3x}}{2} = \infty$

L.H $\frac{\infty}{\infty}$

Revenue, Cost, Profit (15 points)

10. The market research department of a guitar manufacturing company recommends that the company manufacture and market, a new model of electric guitar. After suitable test marketing, the marketing department presents the following demand equation:

$$x = 36{,}000 - 40p\,,$$

where x is the demand at $\$p$ per guitar.

a) Solve the demand equation for p. Find the revenue equation in terms of x.

$40p = 36{,}000 - x$

$\boxed{p = 900 - \frac{1}{40}x}$

$R(x) = xp$

$R(x) = \left(900 - \frac{1}{40}x\right)x$

$\boxed{R(x) = 900x - \frac{1}{40}x^2}$

b) The financial department provides the following cost function:

$$C(x) = 8{,}400 + 275x\,.$$

Find the profit equation in terms of x.

$P(x) = R(x) - C(x)$

$P(x) = 900x - \frac{1}{40}x^2 - 8400 - 275x$

$\boxed{P(x) = -\frac{1}{40}x^2 + 625x - 8400}$

c) Find the marginal profit function.

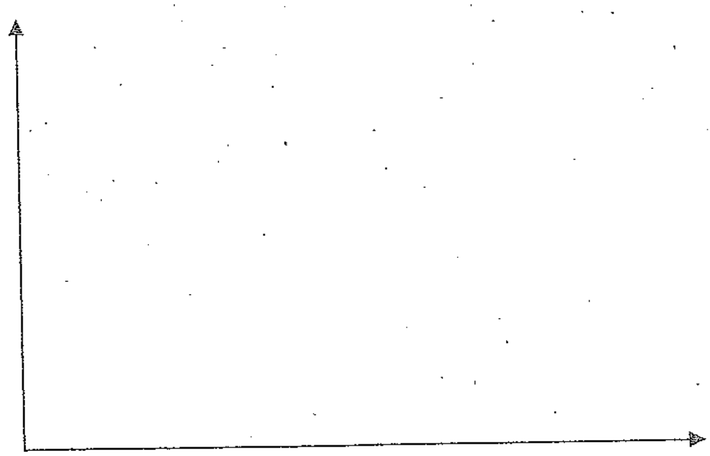
$P'(x) = -\frac{1}{20}x + 625$

d) Find the production level that will produce a maximum profit.

e) Find the maximum profit.

f) Find the price of the new electric guitar that will produce the maximum profit.

g) Sketch the revenue and cost functions below and label areas of profit, areas of loss, and the break-even points.



The Second Derivative (8 points)

11. Find the second derivative of each function:

a) $f(x) = 3x^2 - 5\sqrt{x} - 2x^{-\frac{1}{2}}$

b) $g(x) = \dfrac{4x-2}{5x+3}$