Section 2xx *Complex Numbers*

Imaginary Numbers

What happens if we take the square root of a negative number? For example, what happens when we try to solve an equation like $x^2 + 1 = 0$? Since $x^2 + 1 = 0$, we have $x^2 = -1$ or $x = \pm\sqrt{-1}$. To get a real solution, we would need to have a real number whose square is -1. Since there is no real number whose square is negative, we are left in a quandary - in order to solve this equation we must expand our catalog of numbers. We introduce the complex numbers. Once we have complex numbers available we will be able to solve *all* quadratic equations.

Definition: We define the **imaginary unit** i to be the square root of -1, that is,

$$i = \sqrt{-1}.$$

In order for this definition to make sense, we must have $i^2 = \sqrt{-1}^2 = -1$.

This definition will allow us to describe square roots of negative numbers in terms of i.

Example: Writing Imaginary Numbers

a) $\sqrt{-16} = \sqrt{-1 \cdot 16} = \sqrt{16}\sqrt{-1} = 4i$
b) $\sqrt{-7} = \sqrt{-1 \cdot 7} = \sqrt{-1}\sqrt{7} = i\sqrt{7}$

Exercise: Write the Following in terms of i.
a) $\sqrt{-49}$
b) $\sqrt{-11}$

Note: We usually write the number before the i as in $4i$, but if there is a radical involved we write the radical after as in $i\sqrt{7}$ so it doesn't look like the i is inside the radical.

Definitions: Numbers of the form bi are called **pure imaginary numbers**. **Complex numbers** are numbers of the form $a + bi$. a is a real number called the **real part** and b is a real number called the **imaginary part**. The real numbers are a subset of the complex numbers.

Example: Writing Complex Numbers

a) $5 - \sqrt{-121} = 5 - \sqrt{-1 \cdot 121} = 5 - \sqrt{121}\sqrt{-1} = 5 - 11i$
   5 is the real part and -11 is the imaginary part of this complex number.
b) $2 + \sqrt{-13} = 2 + \sqrt{-1 \cdot 13} = 2 + \sqrt{-1}\sqrt{13} = 2 + i\sqrt{13}$
   2 is the real part and $\sqrt{13}$ is the imaginary part of this complex number.

Exercise: Writing Complex Numbers

a) $5-\sqrt{-49}$
b) $-3+\sqrt{-8}$

Operations with Complex Numbers

Addition of Complex Numbers

To add two complex numbers $a+bi$ and $c+di$, we add the real and imaginary parts:

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

Example: **Adding Complex Numbers**
a) $(1+3i)+3i=(1+0)+(3+3)i=1+6i$
b) $(1+3i)+(2-5i)=(1+2)+(3-5)i=3-2i$
c) $\left(2+\sqrt{-13}\right)+\left(3-\sqrt{-16}\right)=\left(2+i\sqrt{13}\right)+(3-4i)=5+(\sqrt{13}-4)i$

Subtraction of Complex Numbers

To subtract two complex numbers $a+bi$ and $c+di$, we subtract the real and imaginary parts:

$$(a+bi)-(c+di)=(a-c)+(b-d)i$$

Example: **Subtracting Complex Numbers**

a) $(1+3i)-3i=(1-0)+(3-3)i=1$
b) $(1+3i)-(2-5i)=(1-2)+(3+5)i=-1+8i$
c) $\left(2+\sqrt{-13}\right)-\left(3-\sqrt{-16}\right)=\left(2+i\sqrt{13}\right)-(3-4i)=-1-(\sqrt{13}+4)i$

Multiplication of Complex Numbers

To multiply two complex numbers $a+bi$ and $c+di$, we multiply the numbers as if they were two binomials and use the fact that $i^2=-1$ to simplify:

$$\begin{aligned}(a+bi)(c+di)&=ac+adi+bci+bdi^2\\&=(ac-bd)+(ad+bc)i,\ \text{since } i^2=-1\end{aligned}$$

Don't try to memorize this formula - just multiply the two numbers and use the fact that $i^2=-1$ to simplify then combine like terms.

Examples: Multiplying Complex numbers

a) $i(2+2i) = 2i + 2i^2 = -2+2i$

b) $(2+2i)(3-5i) = 2\cdot 3 - 10i + 6i - 10i^2 = 6 - 4i + 10 = 16 - 4i$

Exercise: Operations with Complex Numbers

Perform the indicated operation and write the answer in $a+bi$ form.

a) $(2+2i)+(3-5i)$
b) $(-2-i)+(4-6i)$
c) $(2+2i)-(3-5i)$
d) $i(-4+2i)+(3-4i)$
e) $(-4+2i)(3-4i)$
f) $(3-5i)+(3+4i)(3-4i)$

Division of Complex Numbers

In order to divide complex numbers, we must first be able to find the complex conjugate of a complex number.

Definition: The **complex conjugate** of the complex number $a+bi$ is the number $a-bi$.

Example: The complex conjugate of $2+2i$ is $2-2i$.

Two important properties of complex conjugates are:

1. They are reflections of each other across the real axis in the complex plane.
2. The product of two complex conjugates is a real number, since

$$(a+bi)(a-bi) = a^2 - (bi)^2 = a^2 + b^2 \text{ is a real number.}$$

The second property is particularly useful when dividing complex numbers. If we multiply the denominator by its complex conjugate, then the denominator will become a real number. Of course, we have to multiply the numerator by the same value. This is equivalent to multiplying the fraction by 1.

Example: Dividing Complex Numbers

a) Divide: $\dfrac{1-2i}{3+3i}$

Multiply the numerator and denominator by the complex conjugate of the denominator. You are actually multiplying the fraction by 1. This turns the denominator into a real number. Be sure to write the final answer in $a+bi$ form.

$$\frac{(1-2i)}{(3+3i)}\frac{(3-3i)}{(3-3i)}=\frac{3-9i+6i^2}{3^2-(3i)^2}=\frac{-3-9i}{9+9}=\frac{-3-9i}{18}=-\frac{1}{6}-\frac{1}{2}i$$

To check, we multiply the quotient and the divisor:

$$\left(-\frac{1}{6}-\frac{1}{2}i\right)(3+3i)=-\frac{1}{2}-\frac{1}{2}i-\frac{3}{2}i-\frac{3}{2}i^2=1-2i$$

b) Divide: $\dfrac{3+2i}{5-2i}$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(3+2i)}{(5-2i)}\frac{(5+2i)}{(5+2i)}=\frac{15+16i+4i^2}{5^2-(2i)^2}=\frac{11+16i}{25+4}=\frac{11+16i}{29}=\frac{12}{29}-\frac{16}{29}i$$

c) Divide: $\dfrac{1}{i}$

$$\frac{1}{i}\cdot\frac{-i}{-i}=\frac{-i}{-i^2}=\frac{-i}{1}=-i$$

Powers of i

Evaluate and look for a pattern:

a) i

b) i^2

c) i^3

d) i^4

e) i^5

f) i^7

g) i^8