Chapter 2 The Derivative

Section 1: Tangent Lines

Definition: The **tangent line** to the graph of the function $y = f(x)$ at the point $P(c, f(c))$ is the line that goes through P with slope

$$m_{\text{tangent}} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided that the limit exists. The equation of the tangent line to the graph of the function $y = f(x)$ at the point $P(c, f(c))$ is given by:

$$y = m_{\text{tangent}}(x - c) + f(c)$$

Example 1: Finding Tangent Lines

Find an equation of the tangent line to $y = x^2$ at $P(2, 4)$.

Example 2: Finding Tangent Lines

Find an equation of the tangent line to $y = \dfrac{2}{x}$ at $P(2, 1)$.

Example 3: Finding Tangent Lines

Can you find the tangent line to $y = |x|$ at $x = 0$?

The expression for the slope of the tangent line can also be written as

$$m = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Example 4: Finding Tangent Lines

Find an equation of the tangent line to $y = \sqrt{x}$ at $P(4, 2)$ using the alternative version above.

Average and Instantaneous Velocities

Definition: The average velocity of any object is the change in distance divided by the change in time. Let f be the position function of some object, then the **average velocity** over the interval $(c, c+h)$ is given by

$$\text{Average velocity} = \frac{\text{change in distance}}{\text{change in time}} = \frac{f(c+h)-f(c)}{h}$$

The **velocity** or instantaneous velocity is the limit of the above expression as $h \to 0$:

$$v(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

This means that the velocity of an object at time c, moving with position function f, is given by the slope of the tangent line to f at c.

Example 5: Finding Instantaneous Velocity

Suppose a ball is dropped from a building that is 1080 feet tall.

a) What is the velocity of the ball after 6 seconds?

b) How fast is the velocity of the ball at the instant it hits the ground?

Increments

Suppose x is an independent variable and that y is a dependent variable. Furthermore, suppose y is related to x by the function $y = f(x)$. If x changes from x_1 to x_2, then there is a corresponding change in y. We designate the change in x (called an **increment**) Δx where $\Delta x = x_2 - x_1$. The corresponding change in y is designated Δy where
$\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$.
The difference quotient written in this way is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The difference quotient tells us the average rate of change of y with respect to x over the interval $\left[x_1, x_2\right]$. Geometrically, the difference quotient tells us the slope of the secant line through $(x_1, f(x_1)), (x_2 f(x_2))$. By choosing values of x_2 close to x_1 we can make $\Delta x \to 0$. The limit of the average rates of change is the **instantaneous rate of change of y with respect to** x at $x = x_1$ and is given by

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
$$= \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$