Applications of the Derivative

Increasing and Decreasing Functions

Definition: A function f is **increasing on an interval** (a,b) if $f(x_2) > f(x_1)$ whenever $a < x_1 < x_2 < b$; f is **decreasing on an interval** (a,b) if $f(x_2) < f(x_1)$ whenever $a < x_1 < x_2 < b$.

Theorem: a) If $f'(x) > 0$ on an interval (a,b), then f is increasing on (a,b).

b) If $f'(x) < 0$ on an interval (a,b), then f is decreasing on (a,b).

Example 1: Finding Intervals of Increase and Decrease

Find the intervals where the function $f(x) = 2 - x^3$ is increasing and those for which it is decreasing.

Example 2: Finding Intervals of Increase and Decrease

Find the intervals where the function $f(x) = 2\cos x + \sin^2 x$ is increasing and those for which it is decreasing.

Critical Values and Local Extrema

Definition:

a) If $f(c) \le f(x)$ for all x in some interval (a,b), then $f(c)$ is a **local** (or relative) **minimum** value for the function f.

b) If $f(c) \ge f(x)$ on an interval (a,b), then $f(c)$ is a **local** (or relative) **maximum** value for the function f.

c) The **critical values** (or **critical numbers**) of f are the values of x in the domain of f where $f'(x) = 0$ or $f'(x)$ does not exist.

Theorem 3: Existence of Local Extrema (Fermat's Theorem)

If f is continuous on the interval (a,b) and $f(c)$ is a local extremum, then c is a critical number for the function f.

Example 3: Examine the function $f(x) = x^3$ at $x = 0$. Does the behavior of the graph contradict the theorem on the existence of local extrema?

First-Derivative Test for Local Extrema

Suppose c is a critical number of a continuous function f.

(a) If f' changes sign from negative to positive at $x = c$, then f has a local minimum at c.

(b) If f' changes sign from positive to negative at $x = c$, then f has a local maximum at c.

(c) If f' does not change sign at $x = c$, then f does not have a local minimum or local maximum at c.

Example 4: **Finding Maxima and Minima**

For the function $f(x) = x^3 - 9x^2 + 24x - 10$, find (a) the critical numbers, (b) local maxima and minima, then (c) sketch the graph.

Example 5: **Finding Maxima and Minima**

For the function $f(x) = \frac{x^2}{x^2 - 1}$, find (a) the critical numbers, (b) local maxima and minima, then (c) sketch the graph.

Example 6: **Finding Maxima and Minima**

For the function $f(x) = e^x - x^2$, find (a) the critical numbers, (b) local maxima and minima, then (c) sketch the graph.

Example 7: **Finding Maxima and Minima**

For the functions $f(x) = \sin x + \cos x$, $f(x) = 2x^{1/2} - 4x^{-1/2}$, and $f(x) = x^3 - x$, find (a) the critical numbers, (b) local maxima and minima, then (c) sketch the graph.

Exercise: For the following functions, find (a) the critical numbers, (b) local maxima and minima, but do not sketch the graph.

a) $f(x) = x^3$

b) $f(x) = x^{1/3}$

c) $f(x) = \dfrac{2x^2}{x+3}$

d) $f(x) = \dfrac{x+2}{x-3}$

e) $f(x) = 3 - \dfrac{4}{x} - \dfrac{2}{x^2}$

f) $f(x) = x^3(x-5)$

g) $f(x) = 5x^{1/4} - 2x^{-3/4}$

h) $f(x) = -\dfrac{x^3}{x+1}$

i) $f(x) = |x^2 - 1|$

Section 5: The Second Derivative and Graphs

Definition: **The Second Derivative.** If the derivative f' of a function f itself has a derivative, we call this the second derivative of f and use the symbol f'' to denote this. We can also use write the second derivative of f using Leibniz notation:

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Notation: We can denote the second derivative with a variety of notations, including: $f''(x), \frac{d^2y}{dx^2}, y'', D_x f''(x), D_x^2 f(x)$ Similarly, we can define higher-order derivatives. The **third derivative of a function** $y = f(x)$ is denoted $y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$. In general, the *nth* derivative of a function $y = f(x)$ is denoted $y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$.

Exercise: Find the indicated derivative of the following functions:

1. y'' for $y = 2x^5 - 6x^4 + 2x - 1$
2. $f''(x)$ for $f(x) = \frac{a+bx}{a-bx}$
3. $\frac{d^2y}{dx^2}$ for $y = \frac{1}{x} + 4\sin x$
4. $\frac{d^2y}{dx^2}$ for $y = \frac{\sin x}{x}$
5. $f''(x)$ for $f(x) = \sec x$
6. $\frac{d^2r}{d\theta^2}$ for $r = \cos\theta\sin\theta$
7. $\frac{d^2s}{dt^2}$ for $s = t\csc t$
8. $\frac{d^2r}{d\theta^2}$ for $r = \theta^2\tan\theta$
9. $f'''(x)$ for $f(x) = \sin x\cos x$
10. $f'''(x)$ for $f(x) = \frac{x}{1+x}$

Concavity

Definitions: If $f'(x)$ is increasing on an interval (a,b), then the function f is said to be **concave upward** on (a,b). : If $f'(x)$ is decreasing on an interval (a,b), then the function f is said to be **concave downward** on (a,b).

How do we determine if f' is increasing or decreasing? By looking at the derivative of the derivative, the second derivative, of course.

Test for Concavity: If for each $x \in (a,b)$, we have:

a) $f''(x) > 0$, then $f'(x)$ is increasing on (a,b) and the graph of f is concave upward (smiling or bending to the left).

b) $f''(x) < 0$, then $f'(x)$ is decreasing on (a,b) and the graph of f is concave downward (frowning or bending to the right).

Exercise: Explain what happens when $f''(x) = 0$?

Example 1: The graph of $f(x) = x^3$ changes concavity from concave downward to concave upward at $x = 0$.

Example 2: **Determining Concavity**

Find the intervals where the graph of the function $f(x) = 2x^3 + 9x^2 - 24x - 10$ is concave upward and those intervals for which the graph is concave downward.

Example 3: **Determining Concavity**

Find the intervals where the graph of the function $f(x) = x^4 - 8x^2 + 10$ is concave upward and those intervals for which the graph is concave downward.

Example 4: **Determining Concavity**

Find the intervals where the graph of the function $f(x) = \frac{1-\sqrt{x}}{1+\sqrt{x}}$ is concave upward and those intervals for which the graph is concave downward.

Exercise: **Determining Concavity**

Find the intervals where the graph of the following functions are concave upward and those intervals for which the graph is concave downward.

a) $f(x) = x^4 - 6x^2 + 1$

b) $f(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$

c) $f(x) = \frac{x+3}{x-3}$

Inflection Points

Definition: An **inflection point** is a point on the graph of a function where the concavity changes (from upward to downward or downward to upward). f'' must change sign at this point.

Theorem : If $(c, f(c))$ is an inflection point of the graph of f, then either $f''(c) = 0$ or f'' is undefined at $x = c$.

Exercise: Find the inflection points of the functions given in Examples 1-4 above.

Second-Derivative Test for Local Maxima and Minima

Let f be a function such that $f'(c) = 0$ and the second derivative f'' of f exists on an open interval (a,b) containing c.

a) If $f''(c) > 0$, then $f(c)$ is a relative (local) minimum on (a,b).

b) If $f''(c) < 0$, then $f(c)$ is a relative (local) maximum on (a,b).

c) If $f''(c) = 0$, then the test fails (use the first derivative test).

Example 5: **Finding Local Maxima and Minima**

Find any local maxima, local minima for the function $f(x) = x^2 - \frac{16}{x}$. Find the inflection points and sketch the graph.

Example 6: **Finding Local Maxima and Minima**

Find any local maxima, local minima for the function $f(x) = x + \cos x$. Find the inflection points and sketch the graph.

Example 7: **Finding Local Maxima and Minima**

Find any local maxima, local minima for the function $f(x) = x^{2/3} - 4x^{1/3}$. Find the inflection points and sketch the graph.

Exercise:

a) Find the intervals where the graph of f is concave upward, the intervals where the graph of f is concave downward, and the inflection points for the function $f(x) = x^4 + 24x^2 + 15x - 12$.

b) Find any local maxima, local minima for the function f defined below. Also find the inflection points. Sketch the graph; include the tangent line at each local extreme point.

$$f(x) = x^3 - 9x^2 + 15x + 10.$$

Absolute Maxima and Minima

Definitions:

a) $f(c)$ is an **absolute maximum** of f if $f(c) \geq f(x)$ for every x in the domain of f.

b) $f(c)$ is an **absolute minimum** of f if $f(c) \leq f(x)$ for every x in the domain of f.

Theorem : Continuous function on a closed interval

A continuous function f defined on a closed interval $[a,b]$ has both an absolute maximum and an absolute minimum value.

Steps in Finding Absolute Maximum and Minimum Values

a) Is f continuous over $[a,b]$?

b) Find the values of x for which $f'(x) = 0$ or is undefined (the critical values).

c) Evaluate f at each of the critical values and at the endpoints.

d) The absolute maximum value of $f(x)$ on $[a,b]$ is the largest of the values found in (c).

e) The absolute minimum value of $f(x)$ on $[a,b]$ is the smallest of the values found in (c).

Second-Derivative Test for Absolute Maximum and Minimum Values

Suppose f is continuous on an interval I, and that c is the only critical value of f in I. Then,

a) if $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is an absolute minimum,

b) if $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is an absolute maximum,

c) if $f'(c) = 0$ and $f''(c) = 0$, then the test fails.

Example 8: **Finding Absolute Maximum and Absolute Minimum Values**

Find the absolute maximum and absolute minimum for each of the following functions on the specified intervals, if they exist.

a) $f(x) = x^2 - 4$ for $x \in (-\infty, \infty)$, $x \in (-2,2)$, $x \in [-2,2]$

b) $f(x) = \dfrac{1}{x}$ for $x \in [-3, 3]$, $x \in [1, 3]$.