# Section 10: L'Hopital's Rule

**Theorem 1: (L'Hopital's Rule)** Suppose $f$ and $g$ are functions that are differentiable on the interval $(a,b)$ except possibly at a point $c \in (a,b)$ and that $g'(x) \neq 0$ on $(a,b)$ except possibly at $x = c$. If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right-hand side exists.

**Example 1**: **Using L'Hopital's Rule**

Use L'Hopital's Rule to evaluate the limit: $\lim_{x \to 0} \frac{\sin x}{x}$

**Example 2**: **Using L'Hopital's Rule**

**Us**e L'Hopital's Rule to evaluate the limit: $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

**Example 3**: **Using L'Hopital's Rule**

**Us**e L'Hopital's Rule to evaluate the limit: $\lim_{x \to -1} \frac{x+1}{x^2+4x+3}$

**Example 4**: **Using L'Hopital's Rule**

**Us**e L'Hopital's Rule to evaluate the limit: $\lim_{x \to 1} \frac{\ln x}{x^2}$

**Example 5**: **Using L'Hopital's Rule**

**Us**e L'Hopital's Rule to evaluate the limit: $\lim_{x \to 0} \frac{x^3}{x - \tan x}$

**Example 6**: **Using L'Hopital's Rule**

**Us**e L'Hopital's Rule to evaluate the limit: $\lim_{x\to 0^+} \frac{\sqrt{x}}{\ln(x+1)}$

**Example 7**: **Using L'Hopital's Rule**

**Us**e L'Hopital's Rule to evaluate the limit: $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

**Theorem:** **(L'Hopital's Rule -Version 2)**

Suppose $f$ and $g$ are differentiable on the interval $(a,b)$ except possibly at a point $c \in (a,b)$ and that $g'(x) \neq 0$ on $(a,b)$ except possibly at $x = c$. If $\lim_{x\to c} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and that $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$ (or $\pm\infty$ ). Then,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

**Example 8**: **Using L'Hopital's Rule**

Use L'Hopital's Rule to evaluate the limit: $\lim_{x \to 0} \frac{\tan x - x}{x^3}$

**Example 9**: **Using L'Hopital's Rule**

Use L'Hopital's Rule to evaluate the limit: $\lim_{x \to 0} \frac{1 - \cos x}{\sin x}$

**Example 10**: **Using L'Hopital's Rule**

Use L'Hopital's Rule to evaluate the limit: $\lim_{x \to \infty} \frac{x^2}{e^x}$

**Example 11**: **Using L'Hopital's Rule**

Use L'Hopital's Rule to evaluate the limit: $\lim_{x \to 0^+} \frac{\ln x}{\csc x}$

**Remark**: Some limits of the form $\infty - \infty$, $0^0$, $0 \cdot \infty$, $\infty^0$, and $1^\infty$ can be manipulated into one of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then L'Hopital's Rule can be used to evaluate the limit.

**Example 12**: **Using L'Hopital's Rule**

Use L'Hopital's Rule to evaluate the limit: $\lim_{x \to 0^+} x^x$

**Example 13**: **Using L'Hopital's Rule**

Use L'Hopital's Rule to evaluate the limit: $\lim_{x \to 0} \left[ \frac{1}{\ln(x+1)} - \frac{1}{x} \right]$

**Example 14**: **Using L'Hopital's Rule**

Use L'Hopital's Rule to evaluate the limit: $\lim_{x \to 0^+} (\sin x)^x$