

Section 3 The Derivative as a Function

So far, we have found the derivative of a function at a specific number such as $x = c$. It is also possible to think of the derivative as a function that gives you the slope of the original function when evaluated at a point.

Definition: The **derivative f' of the function f** is the function given by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Geometrically, the function $f'(x)$ gives the slope of the tangent line to the graph of the function f at the point $(x, f(x))$. Other notations for the derivative include the following:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

Definition: A function f is **differentiable** at c if $f'(c)$ exists. A function f is **differentiable on an open interval** (a, b) if $f'(c)$ exists for every $c \in (a, b)$. We can also have differentiability on intervals of these types: (a, ∞) , $(-\infty, a)$, and $(-\infty, \infty)$.

Example 1: Finding the Derivative

Use the definition to find the derivative of the function $f(x) = ax + b$. Then find the equation of the tangent line to the graph that goes through $(1, f(1))$.

Example 2: Finding the Derivative

Use the definition to find the derivative of the function $f(x) = x^3$. Find the equation of the tangent line to the graph of f at the point $(1, f(1))$.

Example 3: Finding the Derivative

Use the definition to find the derivative of the function $f(x) = \frac{1}{x}$.

Theorem: Differentiability and Continuity. If a function f is differentiable at a point $x = c$, then f is continuous at c .

Proof: We use the following (which is true as long as $x \neq c$)

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c}(x - c)$$

Take the limit as $x \rightarrow c$ of both sides using the fact that f is differentiable and the product law:

$$\lim_{x \rightarrow c} [f(x) - f(c)] = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \rightarrow c} (x - c) = f'(c) \cdot 0 = 0$$

Therefore, $\lim_{x \rightarrow c} f(x) = f(c)$ and f is continuous at $x = c$.

There are three ways a function f can fail to be differentiable at a point $x = c$:

1. The graph of f is discontinuous at $x = c$.
2. The graph of f has a vertical tangent line at $x = c$.
3. The graph of f has a "corner" at $x = c$. This means the left and right-hand limits are not the same, so the slope of the curve on the left of $x = c$ is different from the slope of the curve to the right of $x = c$.

Is every continuous function differentiable?

Example 4: Example of a continuous function that is not differentiable

Try to find the slope of $f(x) = \sqrt[3]{x}$ at $x = 0$.

Example 5: Continuity does not imply differentiability

Find the slope of $f(x) = |x|$ on each side of $x = 0$. Use the definition to find the derivative of f at $x = 0$. Is f differentiable for $x = 0$? Is f differentiable for other values of x ?

Higher Derivatives

If a function f has for its derivative the function f' , then it's possible that the derivative might also be differentiable and have a derivative. If it exists, this second derivative is denoted

$$(f')' = f'' = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2}.$$

Example 6: Finding the second derivative

Find the first and second derivatives of $f(x) = x^2 + x$.

If $s = s(t)$ is a position function, then $\frac{ds}{dt} = s'(t) = v(t)$ is a velocity function. The instantaneous rate of change of velocity with respect to time is called **acceleration** and is given by $a(t) = v'(t) = s''(t) = \frac{d^2s}{dt^2}$.

Name _____ Date _____

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Group Work
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