Section 3: Rules of Differentiation

In this section, we will develop the shortcut rules for taking derivatives. We will use the definition to derive several and then use these to further develop

Theorem: Let f and g be functions defined on $[a,b]$ and be differentiable at a point $x \in [a,b]$ and let k be a constant. Then k, kf, $f+g$, $f-g$, fg, and f/g (where $g(x) \neq 0$) are differentiable at x, and

1. The derivative of a constant function $f(x)=k$ is 0.

   Proof: $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\dfrac{k-k}{h} = \lim\limits_{h\to 0}\dfrac{0}{h} = \lim\limits_{h\to 0} 0 = 0$

   Example: If $f(x)=\pi$, then $f'(x)=0$.

2. The derivative of the function kf is $(kf)' = kf'$.

   Proof:
   $$\begin{aligned}\lim_{h\to 0}\frac{kf(x+h)-kf(x)}{h} &= \lim_{h\to 0} k\,\frac{f(x+h)-f(x)}{h}\\ &= k\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = kf'\end{aligned}$$

3. The derivative of the sum function $f+g$ is $(f+g)' = f'+g'$.

   Proof:
   $$\begin{aligned}&\lim_{h\to 0}\frac{\big(f(x+h)+g(x+h)\big)-\big(f(x)+g(x)\big)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}\\ &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right)\\ &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)+\lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right) = f'+g'\end{aligned}$$

4. The derivative of the difference function $f-g$ is $(f-g)' = f'-g'$.

   The proof is similar to the proof of the sum rule.

5. The derivative of the square of a function f^2 is $(f^2)' = 2ff'$.

Proof: $\lim\limits_{h\to 0}\dfrac{\left[f(x+h)\right]^2-\left[f(x)\right]^2}{h}$

$$= \lim_{h\to 0}\frac{\left[f(x+h)+f(x)\right]\left[f(x+h)-f(x)\right]}{h}$$

$$= \lim_{h\to 0}\left[f(x+h)+f(x)\right]\frac{\left[f(x+h)-f(x)\right]}{h}$$

$$= \lim_{h\to 0}\left[f(x+h)+f(x)\right]\lim_{h\to 0}\frac{\left[f(x+h)-f(x)\right]}{h} = 2ff'$$

6. If n is any integer, then $(x^n)' = nx^{n-1}$.

Proof: For this proof, we use the alternative formulation of the derivative $\lim\limits_{c\to x}\dfrac{x^n-c^n}{x-c}$ and use synthetic division.

$$\begin{array}{cccccccc}
 & 1 & 0 & 0 & 0 & 0 & \cdots & -c^n \\
\underline{c} & \_ & \underline{c} & \underline{c^2} & \underline{c^3} & \underline{c^4} & \underline{\cdots} & \underline{c^n} \\
 & 1 & c & c^2 & c^3 & c^4 & \cdots c^{n-1} & 0
\end{array}$$

Therefore, $\lim\limits_{c\to x}\dfrac{x^n-c^n}{x-c} = \lim\limits_{c\to x}\left(x^{n-1}+cx^{n-2}+c^2x^{n-3}+\cdots+c^{n-2}x+c^{n-1}\right)$

$$= x^{n-1}+x\cdot x^{n-2}+x^2\cdot x^{n-3}+\cdots+x^{n-2}x+x^{n-1}$$

$$= \underbrace{x^{n-1}+x^{n-1}+x^{n-1}+\cdots+x^{n-1}+x^{n-1}}_{n\,times} = nx^{n-1}$$

Example 1: Finding derivatives using the shortcut rules

Find the derivative of each of the following functions

a) $f(x) = \dfrac{1}{x^3}$

b) $f(x) = \sqrt{x^5}$

c) $f(x) = 5x^3$

d) $f(x) = 4\sqrt{x^3}$

e) $f(x) = x^5 + 3x^4 - \dfrac{1}{2}x^3 + 2x^2 - 2x + 1$

f) $f(x) = \dfrac{2x^2 - 2x + 1}{\sqrt{x}}$

7. The derivative of the exponential function e^x is $\left(e^x\right)' = e^x$

Proof: $\left(e^x\right)' = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h} = \lim_{h\to 0}\frac{e^x\left(e^h - 1\right)}{h} = e^x \lim_{h\to 0}\frac{e^h - 1}{h}$.

But what is $\lim_{h\to 0}\frac{e^h - 1}{h}$? If we look at the limit numerically, we see that $\lim_{h\to 0}\frac{e^h - 1}{h} = 1$.

It follows that $\left(e^x\right)' = e^x \lim_{h\to 0}\frac{e^h - 1}{h} = e^x \cdot 1 = e^x$.

8. The derivative of the product function fg is $(fg)' = fg' + gf'$.

Proof: We use the formula for the derivative of the square of a function. First, note that $(f+g)^2 = f^2 + 2fg + g^2$. Solving this for fg, we have

$$fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right].$$

Taking the derivative, we get

$$(fg)' = \frac{1}{2}\left[2(f+g)(f'+g') - 2ff' - 2gg'\right]$$

$$= \left[(f+g)(f'+g') - ff' - gg'\right]$$

$$= ff' + fg' + gf' + gg' - ff' - gg' = fg' + gf'$$

Example 2: Finding derivatives using the shortcut rules

Find the derivative of each of the following functions.

a) $f(x) = x^2 e^x$

b) $f(x) = \sqrt{x}\left(1 - x^2\right)$

c) $f(x) = (\sqrt{x} + 2x)\left(4x^2 - \dfrac{2}{x}\right)$

d) Find the derivative in two ways: $f(x) = \left(x^3 + 2x^2 - x + 1\right)\left(3x^2 - x^4 + x - 5\right)$

9. The derivative of the reciprocal of a function f is $(1/f)' = \dfrac{-f'}{f^2}$

Proof: $(1/f)' = \lim\limits_{h\to 0} \dfrac{\dfrac{1}{f(x+h)} - \dfrac{1}{f(x)}}{h}$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x)}{f(x+h)f(x)} - \frac{f(x+h)}{f(x+h)f(x)}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\frac{f(x)-f(x+h)}{f(x+h)f(x)} \\
&= \lim_{h\to 0} \frac{-1}{f(x+h)f(x)}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{-1}{f(x+h)f(x)} \cdot \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \frac{-f'}{f^2}
\end{aligned}
$$

10. The derivative of the quotient function (f/g) is $(\dfrac{f}{g})' = \dfrac{gf' - fg'}{g^2}$.

Proof: $\left(\dfrac{f}{g}\right)' = \left(\dfrac{1}{g}\cdot f\right)' = \dfrac{1}{g}\cdot f' + f\left(\dfrac{-g}{g^2}\right)$ Using the product rule.

Simplifying, we have $\dfrac{1}{g}\cdot f' + f\left(\dfrac{-g}{g^2}\right) = \dfrac{f'}{g} - \dfrac{fg}{g^2} = \dfrac{gf' - fg'}{g^2}$

Example 3: **Finding derivatives using the shortcut rules**

Find the derivative of each function

a) $f(x) = \dfrac{e^x}{1+x}$

b) $f(x) = \dfrac{e^x}{1+x^2}$

c) $f(x) = \dfrac{1+2x}{1-x}$

d) $f(x) = \dfrac{ax+b}{cx+d}$

e) $f(x) = \dfrac{x^2-1}{x^2+1}$

Example 4:

a) Find an equation of the tangent line to the curve $f(x) = \dfrac{\sqrt{x}}{1+x}$ at the point (4,0.4).

b) Find f' and f'' for the function $f(x) = \dfrac{x}{x^2+1}$.

11. The derivative of the sine function is $(\sin x)' = \cos x$.

12. The derivative of the cosine function is $(\cos x)' = -\sin x$.

13. The derivative of the secant function is $(\sec x)' = \sec x \tan x$.

14. The derivative of the cosecant function is $(\csc x)' = -\csc x \cot x$.

15. The derivative of the tangent function is $(\tan x)' = \sec^2 x$.

16. The derivative of the cotangent function is $(\cot x)' = -\csc^2 x$

Proofs:

11. The derivative of the sine function is given by $(\sin x)' = \lim\limits_{h\to 0} \dfrac{\sin(x+h) - \sin x}{h}$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) - \sin(x) + \cos(x)\sin(h)}{h}$$

$$= \lim_{h\to 0} \sin(x)\frac{\cos(h) - 1}{h} + \lim_{h\to 0} \cos(x)\frac{\sin(h)}{h}$$

$$= \sin(x)\cdot \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x)\cdot \lim_{h\to 0} \frac{\sin(h)}{h}$$

In order to find the derivative, we will need to find the two limits:

$\lim\limits_{h\to 0} \dfrac{\cos(h) - 1}{h}$ and $\lim\limits_{h\to 0} \dfrac{\sin(h)}{h}$.

From the figure, we see that:

$$\text{Area } \triangle OAP < \text{Area sector } OAP < \text{Area } \triangle OAT$$

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

If $\theta > 0$, we can multiply be $\dfrac{2}{\sin\theta}$ to get:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Taking reciprocals, we have

$$1 < \frac{\sin\theta}{\theta} < \cos\theta$$

Taking the limit as $\theta \to 0^+$ and using the Squeeze Theorem, we see that $\lim\limits_{\theta\to 0^+} \frac{\sin(\theta)}{\theta} = 1$. A similar argument can be used to show that the limit is the same for $\theta \to 0^-$. Therefore, $\lim\limits_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$.

The second limit we need is $\lim\limits_{h\to 0} \frac{\cos(h)-1}{h}$. To find this, we multiply by the conjugate and simplify:

$$\lim_{h\to 0} \frac{\cos(h)-1}{h}$$

$$= \lim_{h\to 0} \frac{\left(\cos(h)-1\right)\left(\cos(h)+1\right)}{h}$$

$$= \lim_{h\to 0} \frac{\cos^2(h)-1}{h\left(\cos(h)+1\right)}$$

$$= \lim_{h\to 0} \frac{1-\sin^2(h)-1}{h\left(\cos(h)+1\right)} \qquad \text{Since } \cos^2 x = 1-\sin^2 x$$

$$= \lim_{h\to 0} \frac{-\sin^2(h)}{h\left(\cos(h)+1\right)}$$

$$= \lim_{h\to 0} \left(\frac{-\sin(h)}{\left(\cos(h)+1\right)} \right) \cdot \lim_{h\to 0} \frac{\sin(h)}{h} = 0 \cdot 1 = 0$$

Two Important Trigonometric Limits

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1 \text{ and } \lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$$

Now, we can finish finding the derivative:

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$

$$= \sin(x) \cdot \lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h}$$

$$= \sin(x) \cdot o + \cos(x) \cdot 1 = \cos(x)$$

12. The derivative of the cosine function is given by

$$\begin{aligned}
(\cos x)' &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}\\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos x}{h}\\
&= \lim_{h\to 0}\frac{\cos(x)\left[\cos(h)-1\right]-\sin(x)\sin(h)}{h}\\
&= \cos(x)\cdot\lim_{h\to 0}\frac{\left[\cos(h)-1\right]}{h}-\sin(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h}\\
&= \cos(x)\cdot 0-\sin(x)\cdot 1\\
&= -\sin(x)
\end{aligned}$$

13. The derivative of $\sec(x)=\dfrac{1}{\cos(x)}$ is found using the quotient rule:

$$\begin{aligned}
\left[\sec(x)\right]' &= \left[\frac{1}{\cos(x)}\right]'\\
&= \frac{\cos(x)\cdot 1' - 1\cdot\left[\cos(x)\right]'}{\cos^2(x)}\\
&= \frac{\cos(x)\cdot 0 - 1\cdot\left[-\sin(x)\right]}{\cos^2(x)}\\
&= \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x)
\end{aligned}$$

14. The derivative of the cosecant function is given by:

$$(\csc x)' = \left[\frac{1}{\sin(x)}\right]' = \frac{\sin(x)\cdot 0-\cos(x)\cdot 1}{\sin^2(x)} = -\csc x\cot x\,.$$

15. The derivative of the tangent function is given by:

$$(\tan x)' = \left[\frac{\sin(x)}{\cos(x)}\right]' = \frac{\cos(x)\left[\sin(x)\right]'-\sin(x)\left[\cos(x)\right]'}{\cos^2(x)}$$

$$= \frac{\cos(x)\cdot\cos(x) - \sin(x)[-\sin(x)]}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$= \sec^2 x$$

Example 5: Finding derivatives using the differentiation rules

Find the indicated derivatives of the following functions.

1. $f'(x)$ for $f(x) = 3\sin x - \tan x$

2. $\dfrac{dy}{dx}$ for $y = 1 - x\sin x$

3. y' for $y = (\csc x)(\sec x)$

4. $\dfrac{dy}{dx}$ for $y = \dfrac{\sin x}{x}$

5. y' for $y = (x + \cos x)\left(x - \dfrac{1}{x}\right)$

6. $f'(x)$ for $f(x) = \dfrac{x^2 - 4}{x^2 - x - 6}$

7. $\dfrac{dy}{dx}$ for $y = \dfrac{1}{x} + 4 \sin x$

8. $f'(x)$ for $f(x) = \dfrac{\sin x + \cos x}{\cos x}$

9. $f'(x)$ for $f(x) = \dfrac{\cot x}{1 + \cot x}$

10. $\dfrac{dr}{d\theta}$ for $r = \sec\theta\csc\theta$

11. $\dfrac{ds}{dt}$ for $s = t\csc t$

12. $\dfrac{dr}{d\theta}$ for $r = \sec\theta\tan\theta$