Section 7: Derivatives of the Exponential and Logarithmic Functions

Derivative Formulas for $\ln x$ **and** e^x

We can find the derivatives of the natural logarithmic function $f(x) = \ln x$ and the exponential function $f(x) = e^x$ using the definition of the derivative.

The derivative of the natural logarithmic function $f(x) = \ln x$ is given by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\ln(x+h)-\ln x\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\ln\frac{(x+h)}{x}\right]$$

$$= \lim_{h\to 0}\left[\ln(1+\frac{h}{x})\right]^{1/h} \quad \text{Let } m = 1/h$$

$$= \lim_{m\to\infty}\ln\left(1+\frac{x^{-1}}{m}\right)^{m} = \ln e^{x^{-1}} = x^{-1} = \frac{1}{x}.$$

The derivative of the exponential function with base e is found similarly:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$= \lim_{h\to 0} e^x\,\frac{e^h-1}{h} = e^x\lim_{h\to 0}\frac{e^h-1}{h} = e^x\cdot 1 = e^x.$$

(We have assumed that $\lim_{h\to 0}\frac{e^h-1}{h} = 1$).

The derivatives of the natural logarithmic function $f(x) = \ln x$ and the exponential function $f(x) = e^x$ are given by

$$(\ln x)' = \frac{1}{x} \text{ and } (e^x)' = e^x.$$

Exercises:

1. Find the derivative of the following functions:

   a) $f(x) = x \ln x$

   b) $f(x) = \dfrac{\ln x}{x^2}$

   c) $f(x) = xe^x$

   d) $f(x) = x^2 e^x$

   e) $f(x) = e^x \ln(x)$

   f) $f(x) = \dfrac{4e^x}{3x + 1}$

   g) $f(x) = \dfrac{4x^2 + 5}{3e^x}$

h) $f(x) = \left(x^3 \ln x\right)\left(x^3 e^x + x \ln x\right)$