

Chain Rule: General Form

Composite Functions

Definition: Composite function. A function h is a **composite function** or is the **composition** of f and g if it is defined as

$$h(x) = (f \circ g)(x) = f[g(x)]$$

The domain of h is a subset of the domain of g . We must exclude any values of x , for which $f[g(x)]$ is not defined.

Chain Rule

Theorem: The chain rule. If y is a differentiable function of u given by $y = f(u)$ and u is a differentiable function of x given by $u = g(x)$, then $y(x) = f(g(x))$ has a derivative given by

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}, \text{ if both } \frac{dy}{du} \text{ and } \frac{du}{dx} \text{ exist.}$$

This can also be written: $y'(x) = f'(g(x))g'(x)$.

“Proof” We can find the derivative of a composite function from the definition of the derivative in increment form.

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{h(x + \Delta x) - h(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \quad (0.1)$$

Now, note that $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \frac{\Delta u}{\Delta x}$ because a small change in x causes a corresponding change in u and a small change in u causes a corresponding change in y .

Substitute into the right hand of (0.1) to get

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \frac{\Delta u}{\Delta x}$$

We now argue that $\Delta u \rightarrow 0$ as $\Delta x \rightarrow 0$, from which it follows that:

$$\frac{dy}{dx} = \left(\lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} \right) \left(\lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \right) = \frac{dy}{du} \frac{du}{dx}$$

Exercise:

1. This argument is a plausibility argument and not a mathematical proof. Find at least one fault with our argument. Hint: Consider what happens if $\Delta u = 0$ for a nonzero value of Δx .

Generalized Derivative Rules

The following three derivative rules are special cases of the general chain rule.

1. The derivative of a function raised to a power is

$$[(f(x))^n]' = n[f(x)]^{n-1} f'(x) \text{ or } [u^n]' = nu^{n-1} \frac{du}{dx}$$

2. The derivative of the natural logarithm of a function is

$$[\ln(f(x))]' = \frac{1}{f(x)} f'(x) = \frac{f'(x)}{f(x)} \text{ or } [\ln u]' = \frac{1}{u} \frac{du}{dx} = \frac{u'}{u}$$

3. The derivative of the exponential function composed with a function $f(x)$ is

$$(e^{f(x)})' = (e^{f(x)}) f'(x) \text{ or } (e^u)' = (e^u) \frac{du}{dx}$$

Example 1: Using the Chain Rule

Find the derivative $\frac{dy}{dx}$ of the following functions.

a) $y = (x^3 + 3x^2 + 2x + 1)^5$

b) $y = \sqrt{x^2 + 3x + 1}$

c) $y = (5x^3 + 3x^2 - 3)^{5/2}$

d) $y = e^{3x^2+7x-5}$

e) $y = e^{5-\sin x}$

f) $y = [e^{\sin x} + 1]^5$

g) $y = \sqrt{\left(\frac{x^2 + 5}{2x + 3}\right)}$

h) $y = \left[e^{3x^2 + 7x - 5}\right]^2$ Find the derivative two *different* ways.

i) $y = e^{x \cos x}$

j) $y = \sin^2 x e^{\csc x}$

k) $y = \sin(\sin(\sin x))$

To find the derivative of a^x : Use the fact that $a^x = e^{(\ln a)x}$

$$(a^x)' = (e^{(\ln a)x})' = (e^{(\ln a)x}) \ln a = \ln a \cdot a^x$$

Section 9: Implicit Differentiation

Notation

An equation such as $y = 3x + 5$ defines y as a function of x explicitly; x is the independent variable and y is the dependent variable. We could express this relationship using function notation: $y(x) = 3x + 5$.

Examples: Here are examples of other functions defined explicitly:

a) $x = x(t) = 3t^2 + 5t + 7$

b) $z = z(u) = \sqrt{u^3 - 5}$

c) $A = A(r) = \pi r^2$; The area of a circle is a function of the radius.

d) $C = C(r) = 2\pi r$; The circumference of a circle is a function of the radius.

We can extend the function idea to include more general expressions. For example, an equation such as $F(x, y) = 5x^2 + y - 3$ defines a function of two independent variables. For a particular value of $F(x, y)$, say $F(x, y) = 0$, we get an equation such as $5x^2 + y - 3 = 0$. This equation defines y in terms of x implicitly. If we solve for y , we can express the relationship as $y = y(x) = 3 - 5x^2$. This expresses y in terms of x explicitly.

In order to find the derivative, $y' = D_x y$, we think of the equation

$5x^2 + y(x) - 3 = 0$ as defining y implicitly as a function of x and differentiate both sides of the equation with respect to x . This is called implicit differentiation.

In this example it was very easy to solve the equation to find an explicit representation of y in terms of x ; one might be led to consider implicit differentiation a waste of resources. However, there will be implicitly defined functions whose derivatives we seek but whose equations are not solvable explicitly for the variable in question, for example $e^y - y = 5x$.

Implicit Differentiation

Definition: Implicit Differentiation. The process of finding the derivative of a function defined implicitly (such as by an equation) is called **implicit differentiation**.

Examples:

11 Find y' for $y = y(x)$ is given by $5x^2 + y - 3 = 0$.

Implicitly

$$D_x(5x^2 + y - 3) = D_x 0$$

$$D_x(5x^2) + D_x y - D_x(3) = 0$$

$$10x + y' = 0$$

$$y' = -10x$$

Explicitly

$$5x^2 + y - 3 = 0, \text{ Solve for } y.$$

$$y = 3 - 5x^2, \text{ Differentiate}$$

$$y' = -10x$$

11 Find y' for $y = y(x)$ is given by $3x^2 + y - 5 = 0$.

Differentiate both sides with respect to x :

$$D_x[3x^2 + y(x) - 5] = D_x 0$$

$$D_x 3x^2 + D_x y(x) - D_x 5 = 0$$

$$6x + y' - 0 = 0$$

$$y' = -6x$$

11 For y defined implicitly by $F(x, y) = x^2 + y^2 - 49 = 0$. (0.2)

a) Find y' . b) Find the slope of the graph at $x = \sqrt{24}$.

Since $y = y(x)$ is given by $x^2 + [y(x)]^2 - 49 = 0$

Differentiate both sides with respect to x :

$$D_x[x^2 + [y(x)]^2 - 49] = D_x 0$$

$$D_x x^2 + D_x [y(x)]^2 - D_x 49 = 0, \text{ Using the chain rule}$$

$$D_x [y(x)]^2 = 2yy'$$

$$2x + 2yy' - 0 = 0$$

$$y' = -\frac{2x}{2y} = -\frac{x}{y}$$

To find the slope of the graph $x = \sqrt{24}$, after first solving for y , we evaluate

y' for the given value of x and the appropriate value of y : If $x = \sqrt{24}$, then

equation (0.2) gives $(\sqrt{24})^2 + y^2 - 49 = 0$. It follows that $y = \pm 5$

Notation: The symbols $y'|_{(a,b)}$ mean that we are evaluating the function y' at $x = a$ and $y = b$.

If $y' = -\frac{x}{y}$, then the slope of the graph at $x = \sqrt{24}$ is given by the two different expressions $m_1 = y'|_{(\sqrt{24},5)} = -\frac{\sqrt{24}}{5}$, and $m_2 = y'|_{(\sqrt{24},-5)} = -\frac{\sqrt{24}}{-5} = \frac{\sqrt{24}}{5}$.

Exercises:

Find y' , without solving explicitly for y , by using implicit differentiation. Find $y'|_{(x,y)}$ for the given point.

11. $3xy - 2x - 2 = 0$; (2,1)

12. $2x^3y - x^3 + 5 = 0$; (-1,3)

13. $x^2 - y = 4e^y$; (2,0)

14. $\ln y = 2y^2 - x$; (2,1)

15. Find x' for $x = x(t)$ defined implicitly by the equations:

a) $x^3 - tx^2 - 4 = 0$. Evaluate at the point $(t, x) = (-3, -2)$.

b) $t \ln x = xe^t - 1$. Evaluate at the point $(t, x) = (-3, -2)$. ANS. $x'|_{(0,1)} = -1$.

16. Find the equation(s) of the tangent line(s) to the graph of $xy^2 - y - 2 = 0$ at $x = 1$.

17. Find y' for y defined implicitly by the equation $(y - 3)^4 - x = y$. Find the slope of the tangent line to the graph at the point $(-3, 4)$.

18. Find y' for y defined implicitly by the equation $e^{xy} - 2x = y + 1$. Find the slope of the tangent line to the graph at the point $(0, 0)$.

19. Find y' for y defined implicitly by the equation $\sin xy = x^2 - 3$.

20. Find y' for y defined implicitly by the equation $xe^y - 3y \sin x = 1$.

21. Find y' for y defined implicitly by the equation $e^{4y} - \ln y = 2x$.