Section 9 Implicit Differentiation

Notation

An equation such as $y = 3x + 5$ defines $y$ as a function of $x$ explicitly; $x$ is the independent variable and $y$ is the dependent variable. We could express this relationship using function notation: $y(x) = 3x + 5$.

Example 1: Implicitly-Defined Functions

Here are examples of other functions defined explicitly:

a) $x = x(t) = 3t^2 + 5t + 7$

b) $z = z(u) = \sqrt{u^3 - 5}$

c) $A = A(r) = \pi r^2$; The area of a circle is a function of the radius.

d) $C = C(r) = 2\pi r$; The circumference of a circle is a function of the radius.

If we have an equation, such as $5x^2 + y - 3 = 0$, we would say that this equation defines $y$ in terms of $x$ implicitly. If we solve for $y$, we can express the relationship as $y = y(x) = 3 - 5x^2$. This expresses $y$ in terms of $x$ explicitly.

In order to find the derivative, $y' = D_x y$ we think of the equation $5x^2 + y(x) - 3 = 0$ as defining $y$ implicitly as a function of $x$ and differentiate both sides of the equation with respect to $x$. This is called implicit differentiation.

In this example it was very easy to solve the equation to find an explicit representation of $y$ in terms of $x$; one might be led to consider implicit differentiation a waste of resources. However, there will be implicitly defined functions whose derivatives we seek but whose equations are not solvable explicitly for the variable in question, for example $e^y - y = 5x$. 
Implicit Differentiation

**Definition:** The process of finding the derivative of a function defined implicitly (such as by an equation) is called *implicit differentiation*.

**Example 2: Implicit Differentiation**

a) Find $y'$ for $y = y(x)$ is given by $5x^2 + y - 3 = 0$.

<table>
<thead>
<tr>
<th>Implicitly</th>
<th>Explicitly</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_x (5x^2 + y - 3) = D_x 0$</td>
<td>$5x^2 + y - 3 = 0$, Solve for $y$.</td>
</tr>
<tr>
<td>$D_x (5x^2) + D_y y - D_y (3) = 0$</td>
<td>$y = 3 - 5x^2$, Differentiate</td>
</tr>
<tr>
<td>$10x + y' = 0$</td>
<td>$y' = -10x$</td>
</tr>
<tr>
<td>$y' = -10x$</td>
<td></td>
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</tbody>
</table>

**Example 3: Implicit Differentiation**

For $y$ defined implicitly by $F(x, y) = x^2 + y^2 - 49 = 0$.  

a) Find $y'$.  
b) Find the slope of the graph at $x = \sqrt{24}$.

Solution:

Since $y = y(x)$ is given by $x^2 + [y(x)]^2 - 49 = 0$

Differentiate both sides with respect to $x$:

$$D_x \left[ x^2 + [y(x)]^2 - 49 \right] = D_x 0$$

$$D_x x^2 + D_x [y(x)]^2 - D_x 49 = 0$$

Using the chain rule

$$D_x [y(x)]^2 = 2yy'$$

$$2x + 2yy' - 0 = 0$$

$$y' = -\frac{2x}{2y} = -\frac{x}{y}$$

To find the slope of the graph $x = \sqrt{24}$, after first solving for $y$, we evaluate $y'$ for the given value of $x$ and the appropriate value of $y$: If $x = \sqrt{24}$, then equation (1.1) gives $(\sqrt{24})^2 + y^2 - 49 = 0$. It follows that $y = \pm 5$.
If $y' = -\frac{x}{y}$, then the slope of the graph at $x = \sqrt{24}$ is given by the two different expressions:

$$m_1 = y'_{(\sqrt{24},5)} = -\frac{\sqrt{24}}{5}, \quad \text{and} \quad m_2 = y'_{(-\sqrt{24},5)} = -\frac{\sqrt{24}}{-5} = \frac{\sqrt{24}}{5}.$$

**Notation:** The symbols $y'_{(a,b)}$ mean that we are evaluating the function $y'$ at $x = a$ and $y = b$.

**Example 4: Implicit Differentiation**

Find $y'$, without solving explicitly for $y$, by using implicit differentiation. Find $y'_{(x,y)}$ for the given point.

a) $x^2 - y = 4e^y$; \hspace{1cm} (2,0) \\

b) $\ln y = 2y^2 - x$; \hspace{1cm} (2,1)
c) Find the equation(s) of the tangent line(s) to the graph of \(xy^2 - y - 2 = 0\) at \(x = 1\).

d) Find \(y'\) for \(y\) defined implicitly by the equation \(xe'' - 3y\sin x = 1\).

e) Find \(y'\) for the folium of Descartes defined implicitly by the equation \(x^3 + y^3 = 3xy\). Find the equation of the tangent line to the graph at the point \(\left(\frac{3}{2}, \frac{3}{2}\right)\).
Derivatives of Inverse Trigonometric Functions

We can use implicit differentiation to find the derivatives of the inverse trigonometric functions.

**Example 5: Derivatives of Inverse Trigonometric Functions**

Find the derivative of \( y = \cos^{-1} x \).

Let \( y = \cos^{-1} x \), then \( \cos y = x \) for \( 0 \leq y \leq \pi \).

If we take the derivative of both sides using implicit differentiation, we have

\[
\cos y = x
\]

\[
\frac{d}{dx}(\cos y) = \frac{d}{dx}(x)
\]

\[
-\sin y \cdot y' = 1
\]

\[
y' = \frac{1}{-\sin y} = \frac{1}{-\sqrt{1 - \cos^2 y}} = \frac{1}{-\sqrt{1 - x^2}}
\]

Therefore, \( (\cos^{-1} x)' = \frac{1}{-\sqrt{1 - x^2}} \).

**Example 6: Derivatives of Inverse Trigonometric Functions**

Find the derivative of \( y = \tan^{-1} x \).
Derivatives of Logarithmic Functions

Theorem:

- \[ (\ln x)' = \frac{1}{x} \]
- \[ (\log_a x)' = \frac{1}{x \ln a} \]
- \[ (\ln u)' = \frac{1}{u} \]

Proof:

Example 7: Derivatives involving Logarithms

a) \[ (\ln (\sin x))' = \]

b) \[ (\sqrt[3]{\ln x})' = \]

c) \[ (\log_2 (x + \sin x))' = \]
Logarithmic Differentiation

Sometimes it is convenient to find the derivative of an expression by first taking logarithms of both sides and then using implicit differentiation.

**Example 8: Logarithmic Differentiation**

Find $y'$ using logarithmic differentiation: 

$$y = \frac{(x - 3)^3(x + 1)^5}{\sqrt{(2x - 1)^7 \sin(x)}}$$

**Example 9: Logarithmic Differentiation**

Find $y'$ using logarithmic differentiation: 

$$y = x^x$$

**Example 10: Logarithmic Differentiation**

Find $y'$ using logarithmic differentiation: 

$$y^x = x^y$$