# Section 9 Implicit Differentiation

## Notation

An equation such as $y = 3x + 5$ defines $y$ as a function of $x$ explicitly; $x$ is the independent variable and $y$ is the dependent variable. We could express this relationship using function notation: $y(x) = 3x + 5$.

**Example 1: Implicitly-Defined Functions**

Here are examples of other functions defined explicitly:

a) $x = x(t) = 3t^2 + 5t + 7$

b) $z = z(u) = \sqrt{u^3 - 5}$

c) $A = A(r) = \pi r^2$; The area of a circle is a function of the radius.

d) $C = C(r) = 2\pi r$; The circumference of a circle is a function of the radius.

If we have an equation, such as $5x^2 + y - 3 = 0$, we would say that this equation defines $y$ in terms of $x$ implicitly. If we solve for $y$, we can express the relationship as $y = \ y(x) = 3 - 5x^2$. This expresses $y$ in terms of $x$ explicitly.

In order to find the derivative, $y' = D_x y$ we think of the equation $5x^2 + y(x) - 3 = 0$ as defining $y$ implicitly as a function of $x$ and differentiate both sides of the equation with respect to $x$. This is called **implicit differentiation**.

In this example it was very easy to solve the equation to find an explicit representation of $y$ in terms of $x$; one might be led to consider implicit differentiation a waste of resources. However, there will be implicitly defined functions whose derivatives we seek but whose equations are not solvable explicitly for the variable in question, for example $e^y - y = 5x$.

## Implicit Differentiation

**Definition**: The process of finding the derivative of a function defined implicitly (such as by an equation) is called **implicit differentiation**.

### Example 2: Implicit Differentiation

a) Find $y'$ for $y = y(x)$ is given by $5x^2 + y - 3 = 0$.

| Implicitly | Explicitly |
|---|---|
| $D_x\left(5x^2 + y - 3\right) = D_x 0$ | $5x^2 + y - 3 = 0$, Solve for y. |
| $D_x(5x^2) + D_x y - D_x(3) = 0$ | $y = 3 - 5x^2$, Differentiate |
| $10x + y' = 0$ | $y' = -10x$ |
| $y' = -10x$ | |

### Example 3: Implicit Differentiation

For $y$ defined implicitly by $F(x,y) = x^2 + y^2 - 49 = 0$. (1.1)

a) Find $y'$. b) Find the slope of the graph at $x = \sqrt{24}$.

Solution:

Since $y = y(x)$ is given by $x^2 + [y(x)]^2 - 49 = 0$

Differentiate both sides with respect to $x$:

$$D_x\left[x^2 + [y(x)]^2 - 49\right] = D_x 0$$

$$D_x x^2 + D_x [y(x)]^2 - D_x 49 = 0, \quad \text{Using the chain rule}$$

$$D_x [y(x)]^2 = 2yy'$$

$$2x + 2yy' - 0 = 0$$

$$y' = -\frac{2x}{2y} = -\frac{x}{y}$$

To find the slope of the graph $x = \sqrt{24}$, after first solving for $y$, we evaluate $y'$ for the given value of $x$ and the appropriate value of $y$: If $x = \sqrt{24}$, then equation (1.1) gives $\left(\sqrt{24}\right)^2 + y^2 - 49 = 0$. It follows that $y = \pm 5$

If $y' = -\frac{x}{y}$, then the slope of the graph at $x = \sqrt{24}$ is given by the two different expressions $m_1 = y'\big|_{(\sqrt{24},5)} = -\frac{\sqrt{24}}{5}$, and $m_2 = y'\big|_{(\sqrt{24},-5)} = -\frac{\sqrt{24}}{-5} = \frac{\sqrt{24}}{5}$.

**Notation**: The symbols $y'\big|_{(a,b)}$ mean that we are evaluating the function $y'$ at $x = a$ and $y = b$.

## Example 4: Implicit Differentiation

Find $y'$, without solving explicitly for $y$, by using implicit differentiation. Find $y'\big|_{(x,y)}$ for the given point.

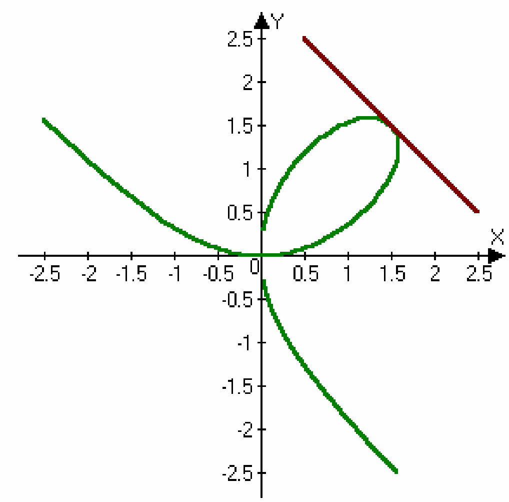
a) $x^2 - y = 4e^y$; $(2,0)$

b) $\ln y = 2y^2 - x$; $(2,1)$

c) Find the equation(s) of the tangent line(s) to the graph of $xy^2 - y - 2 = 0$ at $x = 1$.

d) Find $y'$ for $y$ defined implicitly by the equation $xe^y - 3y\sin x = 1$.

e) Find $y'$ for the folium of Descartes defined implicitly by the equation $x^3 + y^3 = 3xy$. Find the equation of the tangent line to the graph at the point $\left(\dfrac{3}{2}, \dfrac{3}{2}\right)$.



**Derivatives of Inverse Trigonometric Functions**

We can use implicit differentiation to find the derivatives of the inverse trigonometric functions.

**Example 5: Derivatives of Inverse Trigonometric Functions**

**Find the derivative of $y = \cos^{-1} x$.**

Let $y = \cos^{-1} x$, then $\cos y = x$ for $0 \le y \le \pi$.

If we take the derivative of both sides using implicit differentiation, we have

$$\cos y = x$$

$$\frac{d}{dx}(\cos y) = \frac{d}{dx}(x)$$

$$-\sin y \cdot y' = 1$$

$$y' = \frac{1}{-\sin y} = \frac{1}{-\sqrt{1-\cos^2 y}} = \frac{1}{-\sqrt{1-x^2}}$$

Therefore, $\left(\cos^{-1} x\right)' = \dfrac{1}{-\sqrt{1-x^2}}$.

**Example 6: Derivatives of Inverse Trigonometric Functions**

**Find the derivative of $y = \tan^{-1} x$.**

**Derivatives of Logarithmic Functions**

**Theorem:**

- $[\ln x]' = \dfrac{1}{x}$
- $[\log_a x]' = \dfrac{1}{x \ln a}$
- $[\ln u]' = \dfrac{1}{u} u'$

**Proof:**

**Example 7: Derivatives involving Logarithms**

a) $[\ln(\sin x)]' =$

b) $\left[\sqrt[3]{\ln x}\right]' =$

c) $[\log_2 (x + \sin x)]' =$

**Logarithmic Differentiation**

Sometimes it is convenient to find the derivative of an expression by first taking logarithms of both sides and then using implicit differentiation.

**Example 8: Logarithmic Differentiation**

Find $y'$ using logarithmic differentiation: $y = \sqrt{\frac{(x-3)^3(x+1)^5}{(2x-1)^7\sin(x)}}$

**Example 9: Logarithmic Differentiation**

Find $y'$ using logarithmic differentiation: $y = x^x$

**Example 10: Logarithmic Differentiation**

Find $y'$ using logarithmic differentiation: $y^x = x^y$