Increments and Differentials

Increments

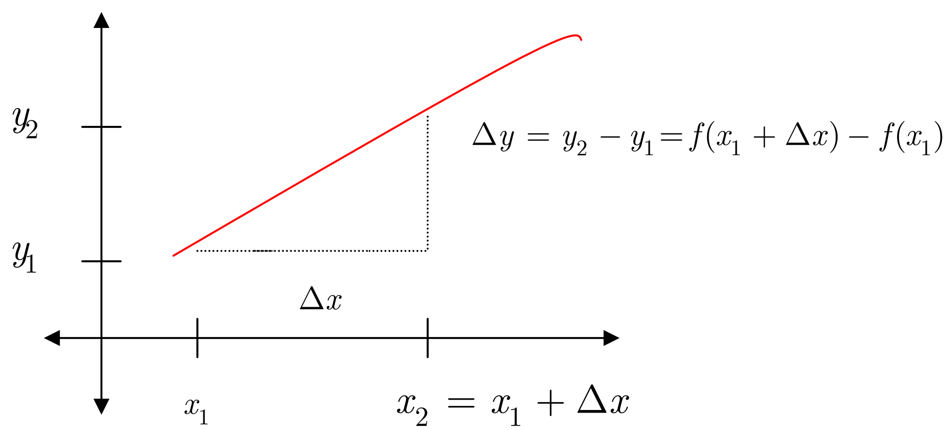
We now ask the question: Given a function defined by the equation $y = f(x)$, what happens to the values of y for small changes in the value of x?

We denote a small change in the independent variable x by Δx (Δx can be positive or negative). If we start with a value of x, say x_1, and we increase this value by a small amount to get x_2, then the difference between the two values is the increment in x:

$$\Delta x = x_2 - x_1.$$

It follows that $x_2 = x_1 + \Delta x$. The change or increment in the dependent variable y is given by

$$\Delta y = y_2 - y_1 = f(x_2) - f(x_1) = f(x_1 + \Delta x) - f(x_1).$$



Exercise 1: Suppose a function f is differentiable on an interval containing the interval (x_1, x_2), explain why $\Delta y \to 0$ as $\Delta x \to 0$.

Example 1: **Finding Δy**

Let f be the square root function $f(x) = \sqrt{x}$. Let $x = 4$ and $\Delta x = 0.3$. Find the difference in y coordinates, Δy.

Here, we have $x = 4$ and $x + \Delta x = 4.3$. The change in y, Δy, is given by

$$\begin{aligned} \Delta y &= f(x + \Delta x) - f(x) \\ &= f(4.3) - f(4) = \sqrt{4.3} - \sqrt{4} \\ &\approx 2.073644 - 2 = 0.073644. \end{aligned}$$

Example 2: **Finding Δy**

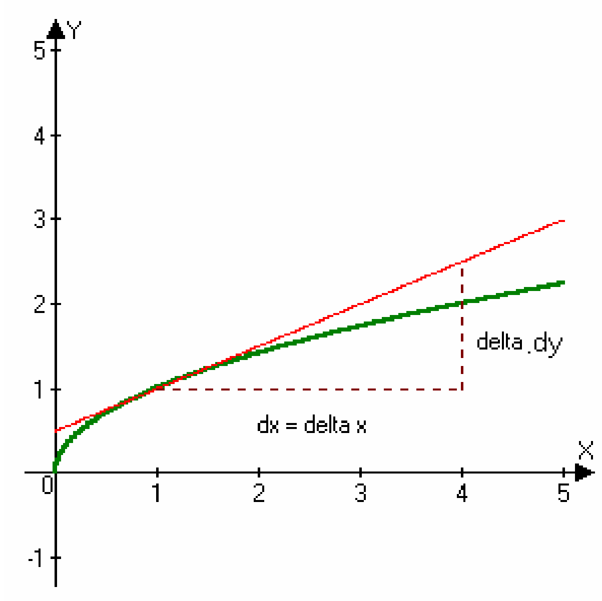
Find Δy for the function $f(x) = x^2$, if $x = 3$ and $\Delta x = 0.1$.

Example 3: **Finding Δy**

Find Δx, Δy, and the ratio $\Delta x / \Delta y$ for the function $f(x) = x^2$, if $x_1 = 1$ and $x_2 = 2$.

Change along the Tangent Line

Recall that the tangent line is the line that most closely matches the graph of the function for values of x close to a given value.



In terms of increments the derivative is given by $f'(x) = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$. If Δx is small then $f'(x) \approx \dfrac{\Delta y}{\Delta x}$ or $\Delta y \approx f'(x)\Delta x$.

Definition: For a differentiable function given by $y = f(x)$, we define the **differential of** f to be $dy = f'(x)dx$.

Example 4: **Comparing Δy and dy**

Find the difference $\Delta y - dy$ for the function $f(x) = \sqrt{x}$ if $x = 4$ and $\Delta x = dx = 0.3$.

Solution:

From example 1, we have $\Delta y = 0.073644$. We also know that $f'(x) = \dfrac{1}{2\sqrt{x}}$ and that $dx = 0.3$. The differential dy is then given by

$$dy = f'(x)dx = \frac{1}{2\sqrt{4}}(0.3) = (0.25)(0.3) = 0.075.$$

The difference $\Delta y - dy$ is

$$\Delta y - dy = 0.073644 - 0.075 = -0.0014.$$

The approximation is accurate to two decimal places of the actual value. The table below shows what happens to the approximation as we take smaller and smaller values of Δx.

$\Delta x = dx$	Δy	dy	$\Delta y - dy$
0.2	0.049390	.05	-0.0006098
0.1	0.024846	.025	-0.00015433
0.01	0.002498	0.0025	-0.000001561
0.001	0.0002499	0.00025	-0.000000016
0.0001	0.00002499	0.000025	-1.56×10^{-10}

We can see at once that the difference between Δy and dy shrinks to zero as Δx shrinks to zero, or $(\Delta y - dy) \to 0$ as $\Delta x \to 0$.

In this example, we have $\frac{dy}{dx} = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. The ratio of differentials $\frac{dy}{dx}$, on the other hand, is $\frac{dy}{dx} = \frac{0.075}{0.3} = \frac{1}{4}$

Exercises:

2. Find the ratio of differentials $\frac{dy}{dx}$ for each of the values in the table in example 4.
3. For a differentiable function $y = f(x)$, what happens to the ratio $\frac{\Delta y}{dy}$ as $\Delta x \to 0$?

4. Find Δy and dy for the following functions if $x = 2$ and Δx is as indicated:

   a) $y = \dfrac{1}{x^2}$, $\Delta x = 0.1$

   b) $y = x^5$, $\Delta x = 0.01$

   c) $y = \sin x$, $\Delta x = 0.1$ and $\Delta x = 0.01$

   d) $y = \csc x$, $\Delta x = 0.1$ and $\Delta x = 0.01$

Estimating Error Using Differentials

We can use the relationship $f(x + \Delta x) \approx f(x) + f'(x)dx$ to approximate the output of a function.

Example 5: Use a differential to approximate $\sqrt{66}$.

We know that $\sqrt{64} = 8$. If we choose $x = 64$, then $\Delta x = 2$ and

$$\sqrt{66} \approx \sqrt{64} + \frac{1}{2\sqrt{64}}(2) = 8 + \frac{1}{2(8)}(2) = 8 + \frac{1}{8} = 8.125.$$

A calculator gives $\sqrt{66} \approx 8.124$. So our approximation is good to two decimal places.

Exercises:

5. Use differentials to approximate the following function values.

   a) $\cos\left(\dfrac{11\pi}{6}\right)$

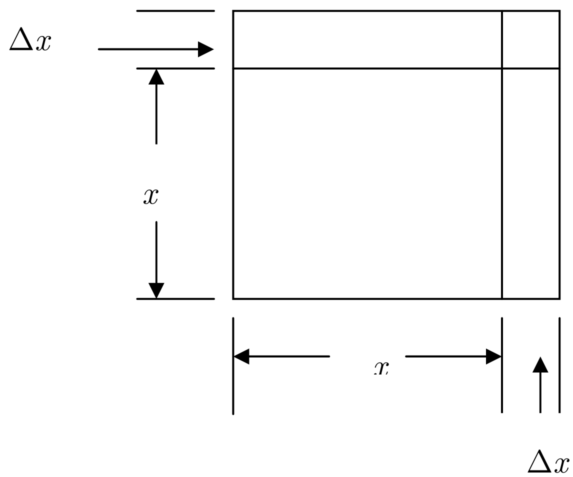
   b) $\sqrt{16.5}$

6. The area of square of side x is given by $A(x) = x^2$.

a) Find dA and ΔA in terms of x and Δx.

b) Shade the part of the square whose area is ΔA.

c) Shade the part of the square in b) whose area is dA.



7. The measurement of the side of a square is found to be 12 inches, with a possible error of $\frac{1}{16}$th of an inch. Use differentials to approximate the error in measuring the area of the square.

to approximate the percent error in the value of g if you can identify the period T to within 0.1 percent of its true value.

8. The measurement of the radius of the end of a log is found to be 14 inches, with a possible error of $\frac{1}{4}$ inch. Use differentials to approximate the error in computing the area of the end of the log.

9. Use differentials to graph the function whose derivative is given by $\frac{dy}{dx} = x + y$ over the interval $0 \le x \le 1$ by using increments $\Delta x = 0.1$. You will need to know that $y = 1$ when $x = 0$.