Calculus is the study of infinite or limiting processes.

Limits

Definition: We write $\lim_{x \to c} f(x) = L$ and say that the limit of $f(x)$ as x approaches c is equal to L. This means we can make the values of $f(x)$ as close to L as we would like by choosing x close to but not equal to c.

Definition: **One–sided limits.** We write $\lim_{x \to c^-} f(x) = L$ and say that the left-hand limit of $f(x)$ as x approaches c is equal to L. This means we can make the values of $f(x)$ as close to L as we would like by choosing x close to but less than c. If x approaches c from above (or from the right), we write $\lim_{x \to c^+} f(x) = L$ and say that the right-hand limit of $f(x)$ as x approaches c is equal to L. This means we can make the values of $f(x)$ as close to L as we would like by choosing x close to but not equal to c.

Theorem: $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$

Theorem 1: Properties of Limits

Suppose that k is a constant and the limits

$\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ both exist. Then the following are true:

1. $\lim_{x \to c}[f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
2. $\lim_{x \to c}[f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$
3. $\lim_{x \to c}[kf(x)] = k \lim_{x \to c} f(x)$
4. $\lim_{x \to c}[f(x)g(x)] = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$
5. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ if $\lim_{x \to c} g(x) \neq 0$
6. $\lim_{x \to c}[f(x)]^n = [\lim_{x \to c} f(x)]^n$
7. $\lim_{x \to c} k = k$
8. $\lim_{x \to c} x = c$
9. $\lim_{x \to c} x^n = c^n$ for any positive integer n
10. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$ for any positive integer n except when n is even we exclude negative values of c.

11. $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$ for any positive integer n except when n is even we assume that $\lim_{x\to c} f(x) \geq 0$.

Theorem 2: If f is a polynomial function, then $\lim_{x\to c} f(x) = f(c)$.

Theorem 3: If f is a rational function and c is in the domain of f, then

$$\lim_{x\to c} f(x) = f(c).$$

Theorem 4: If $f(x) \leq g(x)$ when x is close to c (but not necessarily at c) and the limits of f and g both exist as x approaches c, then

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x).$$

Theorem 5: The Squeeze Theorem. If $f(x) \leq g(x) \leq h(x)$ when x is close to c (but not necessarily at c) and

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$$

then

$$\lim_{x\to c} g(x) = L$$

Continuity

Definitions: A function f is **continuous at a number c** if

$$\lim_{x\to c} f(x) = f(c)$$

If f is not continuous at c, we say that the function is **discontinuous** at c or that the function has a **discontinuity** at c.

Types of Discontinuities

1. **Removable discontinuities** occur when there is a hole in the graph. In this case, the limit exists but the function is either not defined or is defined to be a number different than the limiting value.

2. **Jump discontinuities** occur when the function jumps from one y value to another at a given point. In this case, the limit does not exist because the left and right-hand limits approach different values.
3. **Infinite discontinuities** correspond to vertical asymptotes. In this case, the limit does not exist because one or both of the left or right-hand limits grows without bound.

Definitions: A function f is **continuous from the right** at a number c if $\lim_{x \to c^+} f(x) = f(c)$ and f is **continuous from the left** at a number c $\lim_{x \to c^-} f(x) = f(c)$

Definitions: A function f is **continuous on an interval** (a,b) **if** it is continuous for all $x \in (a,b)$.

Theorem: If f and g are continuous at a point c and k is a constant, then the following functions are continuous at c also:

1. $f + g$
2. $f - g$
3. kf
4. fg
5. $\frac{f}{g}$, If $g(c) \neq 0$
6.

Theorem: Continuity of Polynomial and Rational Functions

(a) Any polynomial is continuous for all $x \in \mathbb{R}$.

(b) Any rational function is continuous for all real numbers x in its domain.

(c) Any root function is continuous for all real numbers x in its domain.

(d) Any trigonometric function is continuous for all x in its domain.

(e) Any inverse trigonometric function is continuous for all x in its domain.

(f) Any exponential function is continuous for all x in its domain.

(g) Any logarithmic function is continuous for all x in its domain.

Theorem: Limit of a Composite Function. If f is continuous at b and $\lim_{x \to c} g(x) = b$, then $\lim_{x \to c} f(g(x)) = f(b)$, or $\lim_{x \to c} f(g(x)) = f(\lim_{x \to c} g(x))$

Theorem: If g is continuous at c and f is continuous at $g(c)$, then

$(f \circ g)(x) = f(g(x))$ is continuous at c.

Theorem: **The Intermediate Value Theorem**. Suppose that f is continuous on the closed interval $[a,b]$ and let k be any number between $f(a)$ and $f(b)$ such that $f(a) < k < f(b)$. Then there exists a number $c \in (a,b)$ such that $f(c) = k$.

Limits and Infinity

Definition: Let f be a function defined on both sides of a real number c, except possibly at c. Then $\lim_{x \to c} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (larger than any number you can choose) by choosing x sufficiently close to but not equal to c.

Definition: The line $x = c$ is called a **vertical asymptote** of the graph of $y = f(x)$ if at least one of the following is true:

$$\lim_{x \to c} f(x) = \infty \qquad \lim_{x \to c^-} f(x) = \infty \qquad \lim_{x \to c^+} f(x) = \infty$$

$$\lim_{x \to c} f(x) = -\infty \qquad \lim_{x \to c^-} f(x) = -\infty \qquad \lim_{x \to c^+} f(x) = -\infty$$

Definition: Let f be a function defined on some interval (a, ∞). Then $\lim_{x \to \infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by choosing x sufficiently large.

Theorem: If n is any positive integer, then $\lim_{x \to \infty} \frac{1}{x^n} = 0$ and $\lim_{x \to -\infty} \frac{1}{x^n} = 0$

Definition: The line $y = L$ is a **horizontal asymptote** for the graph of the function $y = f(x)$ if either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$

Infinite Limits

The notation $\lim_{x \to \infty} f(x) = \infty$ means that the value of $f(x)$ can be made as large as you want by choosing x sufficiently large. Similarly, we can give meaning to the following limits:

$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

Tangent Lines

Definition: The **tangent line** to the graph of the function $y = f(x)$ at the point $P(c, f(c))$ is the line that goes through P with slope

$$m_{\text{tangent}} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided that the limit exists. The equation of the tangent line to the graph of the function $y = f(x)$ at the point $P(c, f(c))$ is given by:

$$y = m_{\text{tangent}}(x - c) + f(c)$$

The expression for the slope of the tangent line can also be written as

$$m = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Average and Instantaneous Velocities

Definition: The average velocity of any object is the change in distance divided by the change in time. Let f be the position function of some object, then the **average velocity** over the interval $(c, c+h)$ is given by

$$\text{Average velocity} = \frac{\text{change in distance}}{\text{change in time}} = \frac{f(c+h) - f(c)}{h}$$

The velocity **or instantaneous velocity is the limit of the above expression as** $h \to 0$**:**

$$v(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

This means that the velocity of an object at time c, moving with position function f, is given by the slope of the tangent line to f at c.

Definition: The **derivative** f' **of the function** f **at** c is given by

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

if the limit exists. We can also write the derivative with the notation

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Facts:

1. The tangent line to $y = f(x)$ at $x = c$ goes through $(c, f(c))$ and has slope $f'(c)$.

2. The derivative $f'(c)$ is the instantaneous rate of change of y with respect to x. That is the derivative $f'(c)$ is the velocity at $x = c$ if $y = f(x)$ is a position function and speed is $|f'(c)|$.

Exercises

1. What is calculus?
2. What is a limit?
3. What does $\lim_{x \to c} f(x) = L$ mean? Use a carefully labeled graph to explain.
4. Explain how to use the Squeeze Theorem. Give an example of its use.
5. Define continuity.
6. Give examples (including graphs) to show how the definition of continuity can fail. i.e. Give examples of the different types of discontinuities.
7. Explain the Intermediate Value Theorem using an appropriately labeled graph.
8. What does $\lim_{x \to \infty} f(x) = L$ mean? Use a graph with labels to explain.
9. What is a vertical asymptote? How do you know if a function has a vertical asymptote at $x = c$.
10. Explain how to find the equation of the tangent line to the function f at the point $x = c$.
11. Explain how to find the instantaneous velocity of the position function f at the point $x = c$.
12. Define the derivative of the function f at the point $x = c$.

Name ____________________ Test 1 Practice for Calc 1 (Learning Lab help is OK)

18. True or False. (Circle one)
 a) If f is a 1-1 function, then $f^{-1}(x)$ exists. T or F
 b) If $x_1 < x_2$ and f is an increasing function, then $f(x_1) < f(x_2)$.
 c) If f is a function with $f(x_1) = f(x_2)$, then $x_1 = x_2$.
 d) If f is a one-to-one function and $f(x_1) = f(x_2)$, then $x_1 = x_2$.
 e) If f is a function with $x_1 = x_2$, then $f(x_1) = f(x_2)$.

19. Give an example of each type of function:
 a) An odd function.
 b) A polynomial that is a one-to-one function.
 c) A transcendental function that has an inverse.
 d) A power function.

20. a) Solve for x exactly. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
 b) Find the inverse function of the original function.
 c) If $f(x) = 2x + \ln x$, find $f^{-1}(2)$

21. A metal container, in the form of a rectangular solid, is to be constructed. The base is to be square, there is to be no top, and the volume of the container is to be 12 cubic meters. Suppose the material for the sides cost $2 per square meter, and the material for the base costs $3 per square meter. Let *C* be the cost of the container, and *L* be the length of a side of the square base. Find a function giving the cost *C* in terms of *L,* and find its domain.

22. Graph $g(x) = 1 - 2\cos\left(3x - \dfrac{\pi}{2}\right)$ (Draw exactly two periods and label the axes carefully.)

23. Solve for x ***exactly***: $2\ln(x+1) - \ln(x+4) = \ln(x-1)$.

24. Suppose that a carton of milk contains 5000 bacteria per cubic inch at the time it was bought, and that the number doubles every day.
 a. Give a function $N(t)$ that gives the number of bacteria *N* in the carton *t* days after the milk was bought.
 b. Find $N(5)$
 c. Suppose that it is not safe to drink the milk when the bacteria count is 3,000,000. For how many days after the carton was bought will it be safe to drink the milk?

9. A string that is 30 inches long is cut into two pieces. One piece will be used to form a square and the other piece to form a circle. Express the total area enclosed by the circle and the square as a function of the perimeter *P* of the square.

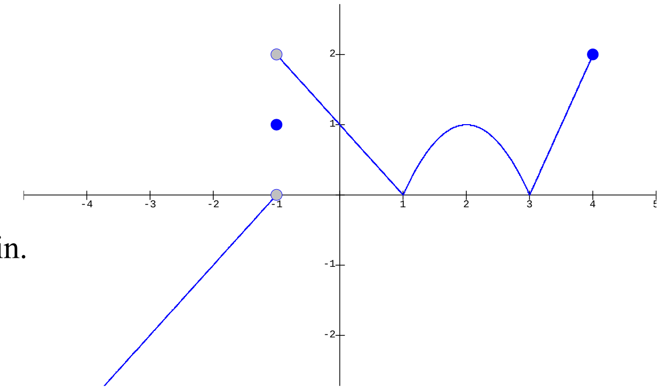
10. Solve $\dfrac{(2+\sin x)(-\sin x)-(\cos x)(\cos x)}{(2+\sin x)^2} = 0$, exactly if possible.
 If not possible, then round your answer to the nearest hundredth.

11. Solve $\cos x - \sin x = 0$, exactly if possible. If not possible, then round your answer to the nearest hundredth.

12. Solve: $x\dfrac{1-2x}{2\sqrt{x-x^2}} + \sqrt{x-x^2} = 0$

13. Using your graphing calculator, find how close x must be to 1 to ensure that $f(x)=\sqrt{x+3}$ is less than 0.1 units from 2.

14. Use the graph of the function $f(x)$ below to answer the following questions:



a) $\lim_{x\to -1^-} f(x)$

b) $\lim_{x\to -1^+} f(x)$

c) $\lim_{x\to -1} f(x)$

d) $f(-1)$

e) Is f continuous at $x=-1$? Explain.

f) $\lim_{x\to 0} f(x)$

g) $\lim_{x\to 3} f(x)$

15. Find the value of the following limits.

a) $\lim_{x\to 6} \frac{x-6}{x+2}$ b) $\lim_{x\to -3} \frac{x^2+x-6}{x+3}$ c) $\lim_{t\to 5} \frac{t^2-25}{2t^2-7t-15}$ d) $\lim_{x\to 10} \frac{\sqrt{x-1}-3}{x-10}$

e) $\lim_{t\to\infty} \frac{2t+4}{t-2}$ f) $\lim_{x\to -\infty} \frac{(x-3)(2x+5)}{(7x-2)(4x+1)}$ g) $\lim_{x\to -\frac{1}{4}^-} \frac{(x-3)(2x+5)}{(7x-2)(4x+1)}$

h) $\lim_{x\to -\frac{1}{4}^+} \frac{(x-3)(2x+5)}{(7x-2)(4x+1)}$ i) $\lim_{x\to -2} \frac{x^2+6x+8}{x^2+7x+10}$ j) $\lim_{x\to\infty} 2e^{-x}$

k) $\lim_{x\to -\infty} 2e^{-x}$ l) $\lim_{x\to\infty} \left(x-\sqrt{x^2-4}\right)$

16. Find values for a and b that will make f continuous everywhere, if

$$f(x)=\begin{cases} 3x+1 & x<2 \\ ax+b & 2\le x<5 \\ x^2 & x\ge 5 \end{cases}$$

17. Suppose $f(x)=\begin{cases} \sqrt{3-x} & x\le 1 \\ x^2 & 1<x<3 \\ \frac{27}{x} & x\ge 3 \end{cases}$.

a. Sketch a graph of the function.

b. $f(-1)=$ ____________________

c. $f(0)=$ ____________________

d. $\lim_{x\to 3^-} f(x)=$ ______________

e. $\lim_{x\to 3^+} f(x)=$ ______________

f. $\lim_{x\to 3} f(x)=$ ______________

g. Is $f(x)$ continuous at $x=3$? Why or why not?

18. If $f(x) = x^2 + 2x - 5$, find and simplify $\dfrac{f(x+h) - f(x)}{h}$. Then, find $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

19. Let $f(x) = \sin x$, to find the following:

   a) The slope M_{PQ} of the secant line through the points $P = \left(\dfrac{\pi}{4}, f\left(\dfrac{\pi}{4}\right)\right)$ and $Q = \left(\dfrac{\pi}{2}, f\left(\dfrac{\pi}{2}\right)\right)$.

   b) Find an approximate equation for the tangent line to $f(x) = \sin x$ at $x = \dfrac{\pi}{4}$.

20. A car is moving as given in the following table

Time(sec)	0	1	2	3	4	5
S (feet)	0	30	50	60	100	150

a. Graph the data.
b. Using the data, compute the *average velocity* of the car on the following time intervals:
   a. [1, 2]
   b. [2, 3]
   c. Estimate the velocity of the projectile when $t = 2$ seconds. State your reasoning.

21. Suppose the height of a projectile fired vertically upward from a height of 1024 feet with an initial velocity of 256 feet per second is given by $h(t) = -16t^2 + 256t + 1024$.
   a) What is the height of the object after 12 seconds?
   b) Find the average velocity over the time period $[12, 12.01]$.
   c) Find the instantaneous velocity at $t = 12$ seconds.

22. Give an example of a function that has a removable discontinuity at $x = 2$, has a horizontal asymptote $y = 2$, and an infinite discontinuity at $x = -2$.

23. Use the Intermediate Value Theorem to show that there is a root of the equation $2\cos x - x^2 = 0$ on the interval $\left[0, \dfrac{\pi}{2}\right]$.

24. Find $\lim\limits_{x \to -\infty} f(x)$ if $\dfrac{2x}{x^2 + 2x + 1} \le f(x) \le \dfrac{2x + 5}{x^2 - 7x - 1}$. Be careful to explain your reasoning.