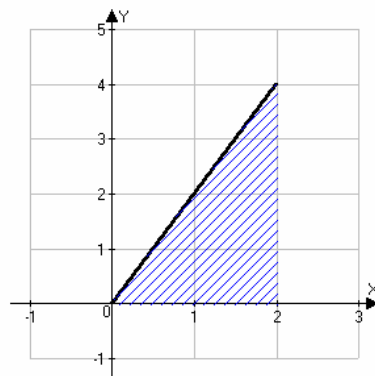


# Chapter 5 Integration

## Section 1: Summation Notation and Approximating Sums

In this section, we examine how to find the exact area under a curve.

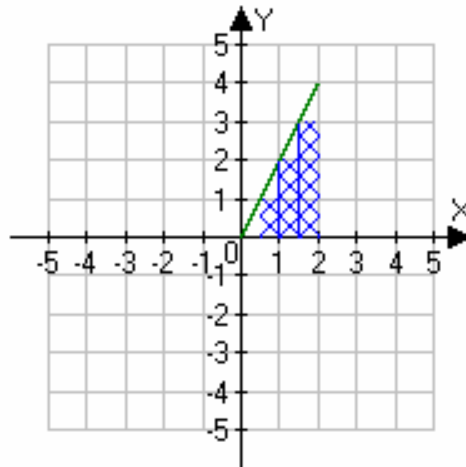
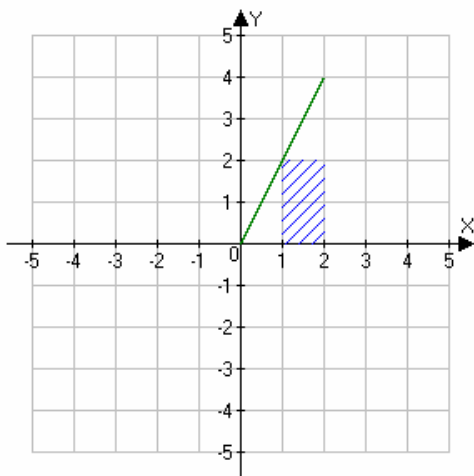
Suppose we wish to find the area under the curve  $f(x) = 2x$  from  $a = 0$  to  $b = 2$ . This area is the shaded region in the figure below. We will do this by approximating the area using rectangles.

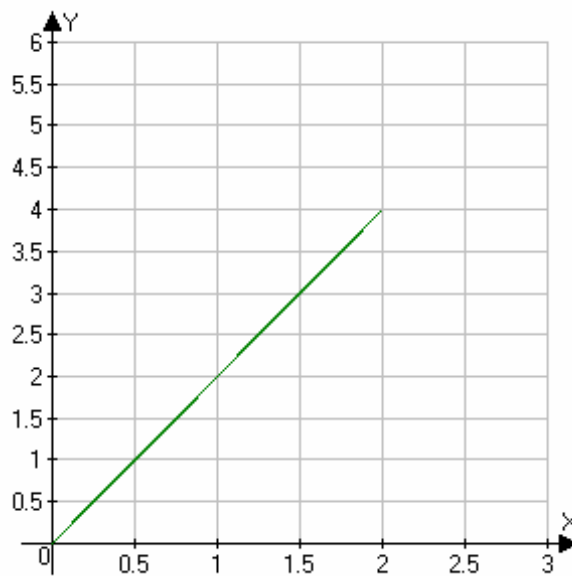


### Example 1: Approximating Areas

Approximate the area under the curve  $f(x) = 2x$  from  $a = 0$  to  $b = 2$  using:

- a) 2 rectangles
- b) 4 rectangles
- c) 8 rectangles





There is no reason to stop this process with 8 rectangles. Our goal is to express the area in terms of  $n$  rectangles and to then let the number  $n$  go to infinity. The resulting limit will give us the exact area.

**Example 2: Approximating Areas**

Express the area under the curve  $f(x) = 2x$  from  $a = 0$  to  $b = 2$  using  $n$  rectangles.

## Summation Notation

In order to evaluate sums of this form, we will first need to discuss several important results. We now introduce and develop a compact notation that is useful for expressing and evaluating sums involving a large number of terms. Consider the sum

$$1 + 2 + 3 + 4 + 5 + 6 + \cdots + (n - 1) + n$$

where  $n$  is any positive integer. In our new notation, this can be written in a convenient manner as

$$1 + 2 + 3 + \cdots + (n - 1) + n = \sum_{i=1}^n i,$$

where  $i$  is called the **index of summation**. The index of summation tells us to begin the sum at a value equal to the **lower limit of summation** (here,  $i = 1$ ) including all the integer values up to and including the **upper limit of summation** (here,  $i = n$ ).

### Example 3: Using Summation Notation

Express the sum  $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16$  using summation notation.

Solution:

$$2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 = \sum_{k=1}^8 2k.$$

**Notation:** Let  $x_1, x_2, x_3, \dots, x_n$  be a set of  $n$  real numbers. The sum  $x_1 + x_2 + x_3 + \cdots + x_n$  will be denoted by the symbol  $\sum_{i=1}^n x_i$  or  $\sum_{i=1}^n x_i$ , which is read “the sum of  $x$  sub  $i$  as  $i$  goes from 1 to  $n$ ”. The specific choice of letter used for the index of summation does not change the value of the sum; we will often use  $i, j,$  and  $k$  interchangeably. The following sums are equal:

$$\sum_{i=1}^n i = \sum_{j=1}^n j = \sum_{k=1}^n k$$

**Example 4: Using Summation Notation**

Express the following sums using summation notation.

a)  $3 + 6 + 9 + 12 + 15 + 18 + 21 + 24$

b)  $(3 \cdot 1 + 1) + (3 \cdot 2 + 1) + (3 \cdot 3 + 1) + \cdots + (3 \cdot n + 1)$

c)  $(2 \cdot 1^2 + 1) + (2 \cdot 2^2 + 1) + \cdots + (2 \cdot n^2 + 1)$

d)  $\left(\frac{1}{1}\right) + \left(\frac{1}{2}\right) + \cdots + \left(\frac{1}{n}\right)$

**Example 5: Evaluating sums**

a)  $\sum_{k=1}^5 3$

c)  $\sum_{k=1}^4 2k^2$

b)  $\sum_{i=1}^7 3i$

d)  $\sum_{k=1}^7 (k^2 + 3k)$

- e) Can you think of a simplification to make when finding the sum in d)?  
Which basic properties of real numbers must you use?

**Theorem: Facts about summation notation**

1.  $\sum_{k=1}^n cx_k = c \sum_{k=1}^n x_k$
2.  $\sum_{k=1}^n (x_k + y_k) = \sum_{k=1}^n x_k + \sum_{i=1}^n y_k$
3.  $\sum_{k=1}^m x_k + \sum_{k=m+1}^n x_k = \sum_{k=1}^n x_k$ , assuming  $1 \leq m < n$ .
4.  $\sum_{k=1}^n c = cn$
5.  $\sum_{k=1}^n k = \frac{n(n+1)}{2}$
6.  $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$
7.  $\sum_{k=1}^n k^3 = \left( \sum_{k=1}^n k \right)^2 = \left( \frac{n(n+1)}{2} \right)^2 = \frac{n^2(n+1)^2}{4}$

**Example 6: Evaluating sums**

Find the following sums:

a)  $\sum_{k=1}^{40} k$

b)  $\sum_{k=1}^{50} 3k$

c)  $\sum_{k=1}^{100} (5k + 2)$

d)  $\sum_{k=1}^{40} (5k^2 + 2)$

$$\text{e) } \sum_{k=1}^n (3k^2 + 2k + 7)$$

$$\text{f) } \sum_{k=1}^n \left( \frac{7k^2}{n^3} + \frac{3k}{n^2} + \frac{2}{n} \right)$$

$$\text{g) } \sum_{k=1}^n \left( \frac{7k^2}{n^2} + \frac{3k}{n} + 2 \right) \left( \frac{3}{n} \right)$$

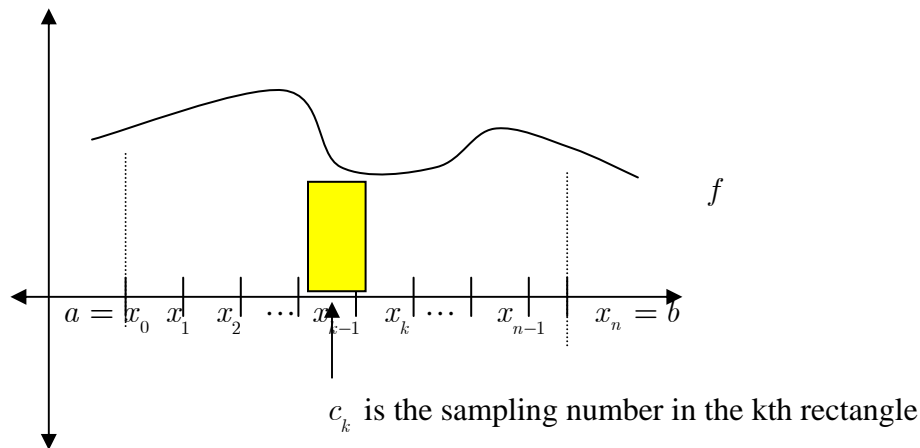
**Example 7: Approximating Areas**

Express the area under the curve  $f(x) = 2x$  from  $a = 0$  to  $b = 2$  using  $n$  rectangles and find a closed form for the sum.

## Sigma Notation for Approximating Sums

The typical approximating sum for any function  $f$  and any interval  $[a, b]$  in the domain of  $f$  can be formed by first partitioning  $[a, b]$  into  $n$  intervals such that

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b.$$



Then a sampling number  $c_k$  is chosen from the  $k$ th interval, the function  $f$  is evaluated at each  $c_k$  and finally the approximating sum is formed:

$$f(c_1)(x_1 - x_0) + \dots + f(c_i)(x_i - x_{i-1}) + \dots + f(c_n)(x_n - x_{n-1})$$

In sigma notation, this is  $\sum_{k=1}^n f(c_k)(x_k - x_{k-1}) = \sum_{k=1}^n f(c_k)\Delta x_k$ , where  $\Delta x_k = x_k - x_{k-1}$ . If

the intervals in the partition are equal, then the width of each rectangle is

$$\Delta x_k = \Delta x = \frac{b - a}{n}.$$

If the intervals are chosen to be equal and the sampling number  $c_k$  is chosen to be the right-hand endpoint of each interval, then  $c_k = a + k\Delta x$ . If the intervals are chosen to be equal and the sampling number  $c_k$  is chosen to be the left-hand endpoint of each interval, then  $c_k = a + (k - 1)\Delta x$ .

**Example 8: Approximating Sums**

Find the approximating sum  $\sum_{k=1}^n f(c_k)\Delta x_k$  for the function  $f(x) = 2x + 5$  on the interval  $[0, 1]$ .

Solution:

Use the right-hand endpoint for the sampling number. Use the properties of summation notation to find a closed-form in terms of  $n$ .

Since  $a = 0$  and  $b = 1$ , the width of each rectangle is

$$\Delta x = \frac{b - a}{n} = \frac{1 - 0}{n} = \frac{1}{n}.$$

The sampling number is given by

$$c_k = a + k\Delta x = 0 + k\frac{1}{n} = \frac{1}{n}k.$$

The height of the  $k$ th rectangle is given by

$$f(c_k) = 2c_k + 5 = 2\left(\frac{1}{n}k\right) + 5 = \frac{2}{n}k + 5.$$

The approximate area is given by the sum of  $n$  rectangles:

$$\sum_{k=1}^n f(c_k)\Delta x_k = \sum_{k=1}^n \left(\frac{2}{n}k + 5\right)\frac{1}{n} = \sum_{k=1}^n \left(\frac{2}{n^2}k + \frac{5}{n}\right).$$

Simplifying the sum, we get

$$\begin{aligned} \sum_{k=1}^n f(c_k)\Delta x_k &= \sum_{k=1}^n \left(\frac{2}{n^2}k + \frac{5}{n}\right) = \sum_{k=1}^n \frac{2}{n^2}k + \sum_{k=1}^n \frac{5}{n} = \frac{2}{n^2} \sum_{k=1}^n k + \frac{1}{n} \sum_{k=1}^n 5 \\ &= \frac{2}{n^2} \frac{n(n+1)}{2} + \frac{1}{n} 5n = \frac{n^2 + n}{n^2} + 5 = \left(1 + \frac{1}{n}\right) + 5 = 6 + \frac{1}{n}. \end{aligned}$$

**Example 9: Approximating Sums**

Find approximating sums for  $f(x) = 2x + 1$  on the interval  $[0, 4]$ , and find a closed-form expression in terms of  $n$  for the sum. Find the exact area by taking the limit.

**Example 10: Approximating Sums**

Find approximating sums for  $f(x) = x^2$  on the interval  $[0, 2]$ , and find a closed-form expression in terms of  $n$  for the sum. Find the exact area by taking the limit.