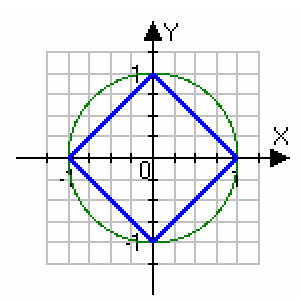
Section 1: The Limit of a Function

In this section, we will consider what happens to the output values of a function as the inputs get close to a certain number. We begin with a geometric example that reduces to the type of problem we consider in this chapter.

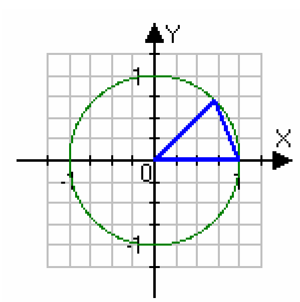
Area of the Unit Circle

We desire to find the area of the unit circle. We begin by estimating the area using an inscribed square. In this case, the area can be thought of as being the sum of eight triangles.



Area of square is $A_4 = \sqrt{2} \cdot \sqrt{2} = 2$

We get a better estimate by doubling the number of triangles and using eight.

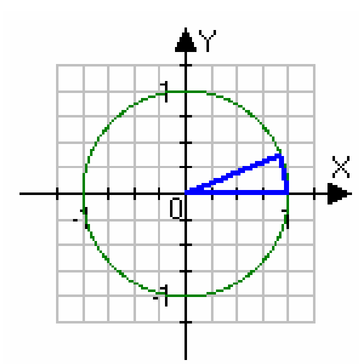


Area of each triangle is $A_T = \frac{1}{2}bh = \frac{1}{2} \cdot 1 \cdot (\sin\theta)$

where $\theta = \frac{2\pi}{8}$

Area of eight triangles = $A_8 = 8\left(\frac{1}{2}\sin\frac{\pi}{4}\right) = 2\sqrt{2}$

We can improve the accuracy of our estimate even more by using sixteen triangles.



Area of each triangle is

$A_T = \frac{1}{2} \cdot 1 \cdot \left(\sin\frac{2\pi}{16}\right)$

Area of sixteen sided polygon =

$A_{16} = 16\left(\frac{1}{2}\sin\frac{2\pi}{16}\right) = 8\sin\frac{\pi}{8}$

In general, if we estimate the area of the circle using $2n$ triangles, then the sum of the areas is given by

$$A_{2n} = 2n\left(\frac{1}{2}\sin\frac{2\pi}{2n}\right) = n\sin\frac{\pi}{n}.$$

We consider what happens as the number of rectangles, n, grows without bound. Does the sum of the areas approach a single number? Clearly, as n increases, the sum of the areas of the triangles must get closer to the area of the unit circle but how can we *prove* this?

We begin with $A_{2n} = n\sin\frac{\pi}{n}$ and let $x = \frac{\pi}{n}$ so that $n = \frac{\pi}{x}$. Then, we have

$$A = \frac{\pi}{x}\sin x = \pi\frac{\sin x}{x}.$$

Notice that as *n* gets large *x* shrinks to 0. In order to find the area, we must determine what happens to $\frac{\sin x}{x}$ as *x* shrinks to 0. This is our first example of a *limit*. We write the statement as $\lim_{x\to 0}\frac{\sin x}{x}$.

Example 1: Examining a Limit Numerically and Graphically

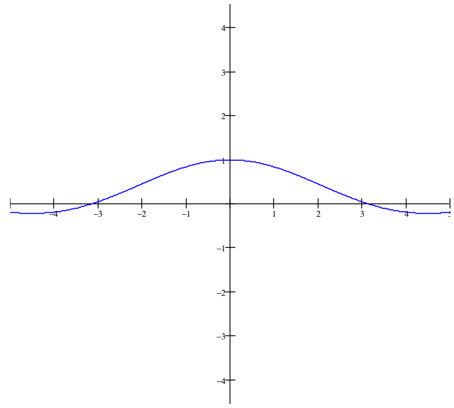
Examine the behavior of the function $f(x) = \frac{\sin x}{x}$ for values of *x* close to, but not equal to, 0. Do this both numerically and graphically.

Solution

Numerically: Fill in the following table to determine what happens to the values of the function as *x* gets close to, but not equal to, 0.

x	$f(x)$
-0.1	
-0.001	
-0.0000001	
0	N/A
0.0000001	
0.00001	
0.001	
0.1	

Graphically: Use the graph to determine what happens to the values of $f(x) = \dfrac{\sin x}{x}$ for x values close to $x = 0$.



Definition: We write $\lim_{x \to c} f(x) = L$ and say that the limit of $f(x)$ as x approaches c is equal to L. This means we can make the values of $f(x)$ as close to L as we would like by choosing x close to but not equal to c.

Remark: Using this notation, the result of Example 1 can be written $\lim_{x \to 0} \dfrac{\sin x}{x} = 0$.

Example 2: Limitations of the Graphical Method

Try to find $\lim_{x \to 0} \sin\left(\dfrac{1}{x^2}\right)$ graphically.

Example 3: Limitations of the Numerical Method

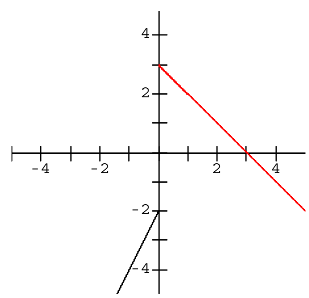
Try to find $\lim_{x\to 0}\left(x^5 + \frac{\cos(17x) - 2e^x}{100,000}\right)$ numerically.

Example 4: Finding One-Sided Limits from a Graph

Examine the behavior of the piecewise-linear function graphed below for values of x close to, but not equal to, 0.

Solution

As *x* approaches 0 from the left side, the values of $f(x)$ get close to -2. As *x* approaches 0 from the right side, the values of $f(x)$ get close to 3. These are examples of one-sided limits.



■

Definition: **One–sided limits.** We write $\lim_{x\to c^-} f(x) = L$ and say that the left-hand limit of $f(x)$ as x approaches c is equal to L. This means we can make the values of $f(x)$ as close to L as we would like by choosing x close to but less than c. If x approaches c from above (or from the right), we write $\lim_{x\to c^+} f(x) = L$ and say that the right-hand limit of $f(x)$ as x approaches c is equal to L. This means we can make the values of $f(x)$ as close to L as we would like by choosing x close to but not equal to c.

Example 5: One-sided limits

For the function in Example 4, find

a) $\lim_{x \to 0^-} f(x)$

b) $\lim_{x \to 0^+} f(x)$

Solution

The left-hand limit at $x = 0$ is $\lim_{x \to 0^-} f(x) = -2$. The right-hand limit at $x = 0$ is $\lim_{x \to 0^+} f(x) = 3$.

Since the function values approach two different values as we approach 0 from the left and from the right, we will say that the function does not have a limit at $x = 0$.

■

Theorem: $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$

Example 6: Evaluating Limits Graphically

Find the indicated limits, if they exist. If they do not exist, explain why they do not.

a) $\lim_{x \to -2} f(x)$

b) $\lim_{x \to 0} f(x)$

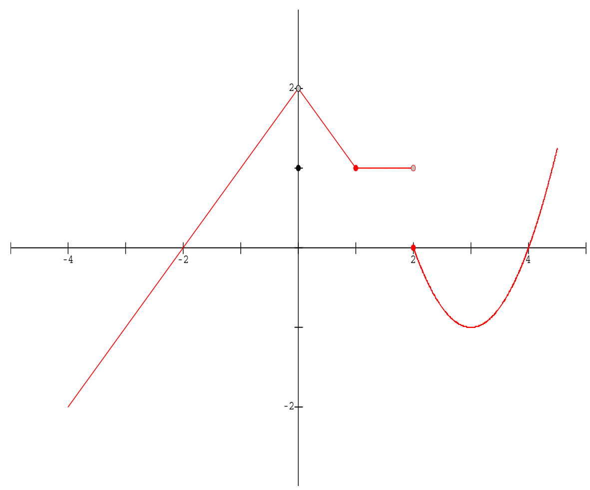
c) $\lim_{x \to 1} f(x)$

d) $\lim_{x \to 2^-} f(x)$

e) $\lim_{x \to 2^+} f(x)$

f) $\lim_{x \to 2} f(x)$

g) $\lim_{x \to 3} f(x)$



Example 7: Evaluating One-sided Limits

Examine the following limits by looking at the left-hand and right-hand limits.

a) $\lim_{x \to 0} \frac{1}{x}$

b) $\lim_{x \to 0} \frac{|x|}{x}$

c) $\lim_{x \to 0} \frac{1}{x^2}$

Example 8: Calculating Limits

Calculate the following limits for each of the given functions:

i. $\lim_{x \to 2^-} f(x)$

ii. $\lim_{x \to 2^+} f(x)$

iii. $\lim_{x \to 2} f(x)$

a) $f(x) = \frac{|x-2|}{x-2}$

b) $f(x) = \frac{x^2-4}{x-2}$

c) $f(x) = \frac{x-2}{x^2-4}$

d) $f(x) = \frac{1}{(x-2)^2}$

Example 9: Calculating Limits Algebraically

Find the following limits.

a) $\lim_{x \to -2} \frac{2x+4}{x^2-4}$

b) $\lim_{t \to 0} \frac{\sqrt{4+t}-2}{t}$