Section 2: Properties of Limits

In order to simplify our work in evaluating limits, we introduce a set of rules that can be used to evaluate limits.

Theorem 1: Properties of Limits

Suppose that k is a constant and the limits

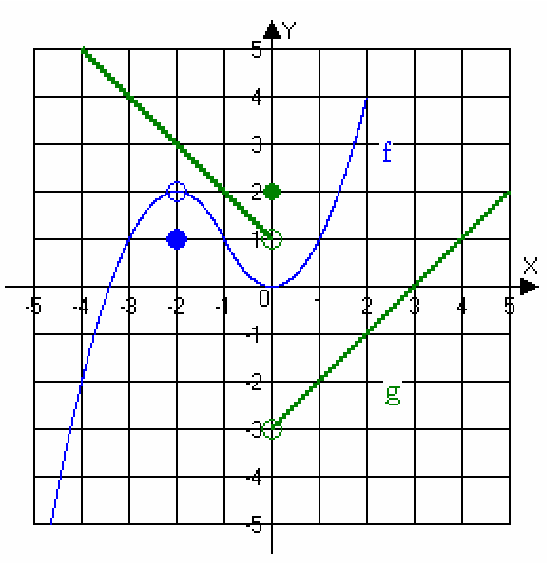
$\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist. Then the following are true:

1. $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$ Example: $\lim_{x\to 0}(\sin x + \cos x)$
2. $\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$ Example: $\lim_{x\to 0}(\sin x - \cos x)$
3. $\lim_{x\to c}[kf(x)] = k\lim_{x\to c} f(x)$ Example: $\lim_{x\to \frac{\pi}{2}}(2\sin x)$
4. $\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x)\lim_{x\to c} g(x)$ Example: $\lim_{x\to \frac{\pi}{2}}(\sin x\cos x)$
5. $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$ Example: $\lim_{x\to 2}\dfrac{x+2}{x+4}$
6. $\lim_{x\to c}[f(x)]^n = [\lim_{x\to c} f(x)]^n$ Example: $\lim_{x\to \pi}(\cos^5 x)$
7. $\lim_{x\to c} k = k$ Example: $\lim_{x\to 5}\pi$
8. $\lim_{x\to c} x = c$ Example: $\lim_{x\to 5} x$
9. $\lim_{x\to c} x^n = c^n$ for any positive integer n Example: $\lim_{x\to 2} x^3$
10. $\lim_{x\to c}\sqrt[n]{x} = \sqrt[n]{c}$ for any positive integer n except when n is even we exclude negative values of c. Example: $\lim_{x\to 4}\sqrt{x}$
11. $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$ for any positive integer n except when n is even we assume that $\lim_{x\to c} f(x) \geq 0$.

Example: $\lim_{x\to \frac{\pi}{2}}\sqrt[4]{\sin x} = \sqrt[4]{\lim_{x\to \frac{\pi}{2}}\sin x}$

Example 1: Using the Properties of Limits

Use the graphs of the function f and g given below and the properties of limits to find the indicated limits.



a) $\lim_{x \to -2}[f(x) - 3g(x)]$

b) $\lim_{x \to 0}[f(x)g(x)]$

c) $\lim_{x \to 1}\left[\frac{f(x)}{g(x)}\right]$

d) $\lim_{x \to 3}\left[\frac{f(x)}{g(x)}\right]$

Example 2: Calculating Limits Using the Properties of Limits

Find $\lim_{x \to 3} \left(x^2 - x - 6 \right)$ carefully justifying each step.

Example 3: Calculating Limits Using the Properties of Limits

Find $\lim_{x \to 3} \dfrac{x^2 - x - 6}{x - 3}$

Example 4: Calculating Limits Using the Properties of Limits

Find $\lim_{x \to 3} \dfrac{x^2 + x}{x^2 - 3x}$

Example 5: Calculating Limits Using the Properties of Limits

Find $\lim_{x \to 0} \dfrac{x\left(e^{-3x-1}\right)}{x^2 - x}$

Example 6: Calculating Limits Using the Properties of Limits

Find $\lim_{x \to 0} \frac{\sin x}{\cos x}$

Example 7: Calculating Limits Using the Properties of Limits

Find $\lim_{x \to 7} \sqrt{2x^2 - 5x}$

Example 8: Calculating Limits Using the Properties of Limits

Find $\lim_{h \to 0} \frac{(4+h)^2 - 16}{h}$

Example 9: Calculating Limits Using the Properties of Limits

Find $\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$

Theorem 2: If f is a polynomial function, then $\lim_{x \to c} f(x) = f(c)$.

Proof: This is left as an exercise. Just use the properties in Theorem 1 to the polynomial function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$.

Example 10: Finding the Limit of a Polynomial Function

Find $\lim_{x \to 3} \left(2x^3 + 4x^2 - 5\right)$

Theorem 3: If f is a rational function and c is in the domain of f, then

$$\lim_{x \to c} f(x) = f(c).$$

Example 11: Finding the Limit of a Rational Function

Find $\lim_{x \to 3} \frac{x^2 - 5}{2x^3 + 4}$

Theorem 4: If $f(x) \le g(x)$ when x is close to c (but not necessarily at c) and the limits of f and g both exist as x approaches c, then

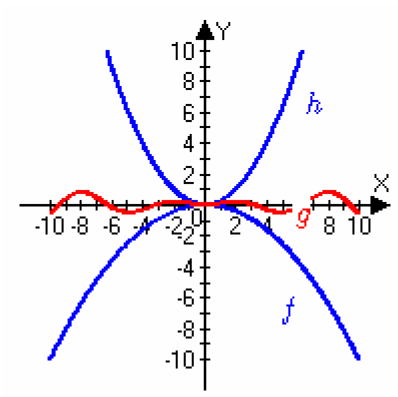
$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x).$$

Theorem 5: The Squeeze Theorem. If $f(x) \le g(x) \le h(x)$ when x is close to c (but not necessarily at c) and

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$$

then

$$\lim_{x \to c} g(x) = L$$



The graph shows the functions *f*, *g*, and *h*. As $x \to 0$ both functions *f* and *h* go to 0. Since the values of the function *g* are trapped between the values of these two functions and they both go to 0, the squeeze theorem guarantees than *g* must also go to 0.

Example 12: Using the Squeeze Theorem

Use the squeeze theorem to find the limit: $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$

Example 13: **Using the Squeeze Theorem**

Use the squeeze theorem to find the limit: $\lim_{x\to 0}\left[\sqrt{x}\cos^2\left(\frac{1}{x}\right)\right]$

Example 14: **Finding Limits of Piecewise-defined Functions**

Find the indicated limits.

a) Find $\lim_{x\to 2^+} f(x), \lim_{x\to 2^-} f(x), \lim_{x\to 2} f(x)$ if $f(x) = \begin{cases} 2x-3 & x \le 2 \\ -x^3+4x+1 & x > 2 \end{cases}$

b) Find $\lim_{x\to 1^+} f(x), \lim_{x\to 1^-} f(x), \lim_{x\to 1} f(x)$ if $f(x)=\begin{cases} 2x^2-3 & x\le 1 \\ -x^2+2x-1 & x>1 \end{cases}$

c) Find $\lim_{x\to 0} f(x)$ if $f(x)=\begin{cases} 2x\cos x & x<0 \\ x^3+2x & x\ge 0 \end{cases}$