

Section 3: Continuity

Definitions: A function f is **continuous at a number c** if

$$\lim_{x \rightarrow c} f(x) = f(c)$$

If f is not continuous at c , we say that the function is **discontinuous** at c or that the function has a **discontinuity** at c .

Example 1: Showing a Function is Discontinuous

Show that the function $f(x) = \begin{cases} \frac{x^2 - 4}{x + 2} & x \neq -2 \\ 0 & x = -2 \end{cases}$ is discontinuous at $x = -2$.

Solution

To see this, we find $\lim_{x \rightarrow -2} f(x)$.

Types of Discontinuities: There are three categories of discontinuities: removable discontinuities, jump discontinuities, and infinite discontinuities.

1. **Removable discontinuities** occur when there is a hole in the graph. In this case, the limit exists but the function is either not defined or is defined to be a number different than the limiting value.

Example: $f(x) = \begin{cases} \frac{x^2 - 4}{x + 2} & x \neq -2 \\ 0 & x = -2 \end{cases}$

2. **Jump discontinuities** occur when the function jumps from one y value to another at a given point. In this case, the limit does not exist because the left and right-hand limits approach different values.

Example: $f(x) = \frac{|x|}{x}$ has a jump discontinuity at $x = 0$.

3. **Infinite discontinuities** correspond to vertical asymptotes. In this case, the limit does not exist because one or both of the left or right-hand limits grows without bound.

Example: $f(x) = \frac{x-2}{x^2-4}$ has a removable discontinuity at $x = 2$ and an infinite discontinuity at $x = -2$.

Definitions: A function f is **continuous from the right** at a number c if $\lim_{x \rightarrow c^+} f(x) = f(c)$ and f is **continuous from the left** at a number c if $\lim_{x \rightarrow c^-} f(x) = f(c)$

Definitions: A function f is **continuous on an interval** (a, b) if it is continuous for all $x \in (a, b)$.

We can also define continuity on an interval, which includes one or both endpoints by using the idea of continuity from the left and right-hand sides. For example, a function f is continuous on an interval $[a, b)$ if it is continuous for all $x \in (a, b)$ and continuous from the right at a .

Theorem: If f and g are continuous at a point c and k is a constant, then the following functions are continuous at c also:

1. $f + g$
2. $f - g$
3. kf
4. fg
5. $\frac{f}{g}$, If $g(c) \neq 0$

Theorem: Continuity of Polynomial and Rational Functions

- (a) Any polynomial is continuous for all $x \in \mathbb{R}$.
- (b) Any rational function is continuous for all real numbers x in its domain.
- (c) Any root function is continuous for all real numbers x in its domain.
- (d) Any trigonometric function is continuous for all x in its domain.
- (e) Any inverse trigonometric function is continuous for all x in its domain.
- (f) Any exponential function is continuous for all x in its domain.
- (g) Any logarithmic function is continuous for all x in its domain.

Theorem: Limit of a Composite Function. If f is continuous at b

and $\lim_{x \rightarrow c} g(x) = b$, then $\lim_{x \rightarrow c} f(g(x)) = f(b)$, or

$$\lim_{x \rightarrow c} f(g(x)) = f(\lim_{x \rightarrow c} g(x))$$

Example 2: Finding the Limit of a Composite Function

Find the value of $\lim_{x \rightarrow 1} \arccos\left(\frac{1 - \sqrt{x}}{1 - x}\right)$

Theorem: If g is continuous at c and f is continuous at $g(c)$, then

$$(f \circ g)(x) = f(g(x))$$

is continuous at c .

Example 3: Functions and Continuity

Where are the following functions continuous? For each of the functions that are discontinuous, discuss the type of discontinuity.

a) $f(x) = \cos(x^2 - 3x + 5)$

b) $g(x) = \frac{\cos(3x + 5)}{e^x - 4}$

c) $h(x) = \ln(x - 3)$

d) $F(x) = \log(x^2 - 3)$

e) $G(x) = (x - 3)^{\frac{3}{2}}$

Theorem: The Intermediate Value Theorem. Suppose that f is continuous on the closed interval $[a,b]$ and let k be any number between $f(a)$ and $f(b)$ such that $f(a) < k < f(b)$. Then there exists a number $c \in (a,b)$ such that $f(c) = k$.

Example 4: Using the Intermediate Value Theorem

Show that the function $x^3 + 2x^2 - 42 = 0$ has a root on the interval $(0,3)$.

Example 5: Using the Intermediate Value Theorem

Show that the equation $\cos x = x$ has at least one solution.