Section 4: Limits and Infinity

Definition: Let f be a function defined on both sides of a real number c, except possibly at c. Then

$$\lim_{x \to c} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (larger than any number you can choose) by choosing x sufficiently close to but not equal to c.

Example 1: Limits that go to infinity

Find the following limits:

a) $\lim_{x \to 0} \frac{1}{x^4}$ b) $\lim_{x \to 0^-} \frac{1}{x^3}$ c) $\lim_{x \to 0^+} \frac{1}{x^3}$ d) $\lim_{x \to 0} \frac{1}{x^3}$

Definition: The line $x = c$ is called a **vertical asymptote** of the graph of $y = f(x)$ if at least one of the following is true:

$\lim_{x \to c} f(x) = \infty$ $\lim_{x \to c^-} f(x) = \infty$ $\lim_{x \to c^+} f(x) = \infty$

$\lim_{x \to c} f(x) = -\infty$ $\lim_{x \to c^-} f(x) = -\infty$ $\lim_{x \to c^+} f(x) = -\infty$

Example 2: Limits that go to infinity

Find the following limits.

a) $\lim_{x \to 2^-} \frac{1}{x-2}$

b) $\lim_{x \to 2^+} \frac{x+1}{x-2}$

c) $\lim\limits_{x \to \frac{\pi}{2}^+} \tan x$

d) $\lim\limits_{x \to \frac{\pi}{2}^-} \tan x$

e) $\lim\limits_{x \to 0^+} \ln x$

Definition: Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by choosing x sufficiently large.

Theorem: If n is any positive integer, then $\lim\limits_{x \to \infty} \frac{1}{x^n} = 0$ and $\lim\limits_{x \to -\infty} \frac{1}{x^n} = 0$

Example 3: Limits as x goes to infinity

Find $\lim\limits_{x \to \infty} \frac{x+1}{x-2}$

Example 4: Limits as x goes to infinity

Find $\lim_{x\to\infty} \sqrt{x(x+1)} - x$

Definition: The line $y = L$ is a **horizontal asymptote** for the graph of the function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L$$

Example 5: A function with two horizontal asymptotes

Show that the function $f(x) = \tan^{-1} x$ has two vertical asymptotes.

Solution

$\lim_{x\to-\infty} \tan^{-1} x = -\dfrac{\pi}{2}$ and $\lim_{x\to\infty} \tan^{-1} x = \dfrac{\pi}{2}$, so the function $f(x) = \tan^{-1} x$ has two horizontal asymptotes.

Fact: $\lim_{x\to-\infty} e^x = 0$, so the x-axis is a horizontal asymptote for the exponential function $f(x) = e^x$. Note also that $\lim_{x\to\infty} e^{-x} = 0$, so the x-axis is a horizontal asymptote for the exponential function $f(x) = e^{-x}$.

Example 6: Limits as x goes to infinity

Find $\lim_{x \to \infty} \frac{5x^2 + 2x + 7}{3x^2 - 5x + 4}$

Infinite Limits

The notation $\lim_{x \to \infty} f(x) = \infty$ means that the value of $f(x)$ can be made as large as you want by choosing x sufficiently large. Similarly, we can give meaning to the following limits:

$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

Example 7: Infinite Limits

a) $\lim_{x \to \infty} e^{2x}$

b) $\lim_{x \to \infty} \ln 2x$

c) $\lim_{x \to -\infty} \left[e^{4x} \sin x \right]$

d) $\lim_{x \to \infty} \frac{-x}{\sqrt{4 + x^2}}$