Section 2: The Definite Integral

In this section, we investigate what happens to the values of an approximating sum when we let the number of intervals n in the partition go to infinity.

Definition: The **mesh of a partition** is the length of the longest section (or sections) in the partition. That is, the mesh of any partition is the largest value of $x_k - x_{k-1} = \Delta x_k$ over all possible values of k.

The Definite Integral

Definition: If f is a function defined on an interval $[a,b]$ and the sums $\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$ approach a certain number as the mesh of the partitions of $[a,b]$ shrink towards 0 (regardless of the choice of c_k in each interval $[x_{k-1}, x_k]$, that number is called the **definite integral** of f over $[a,b]$.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

The sum on the right-hand side is called a **Reimann sum**.

Theorem: Existence of the definite integral. Let f be a continuous function over an interval $[a,b]$. Then the approximating sums $\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$ approach a single number as the mesh of the partitions of $[a,b]$ approaches 0. Hence $\int_a^b f(x)dx$ exists.

Example 1: Writing Reimann sums as integrals

Write the limit $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{1+\left(x_k\right)^2}\Delta x_k$ as a definite integral on the interval $\left[0,\frac{\pi}{2}\right]$.

Example 2: Evaluating a definite integral

Find the definite integral of the function $f(x)=2x+5$ over the interval $[0,1]$.

Example 3: Evaluating a definite integral

Use right-hand endpoints to express $\int_0^2 (2 - x^2)\,dx$ as a Reimann sum and then evaluate.

Example 4: Writing an integral as a Reimann sum

Express the integral $\int_0^{\pi} \sin x dx$ as a Reimann sum. Use a calculator to approximate the sum of the first 50 terms.

Example 5: Evaluating integrals by interpreting as areas

Evaluate $\int_{-2}^{2}\sqrt{4-x^2}\,dx$ by interpreting in terms of an area.

The Midpoint Rule

Theorem: Midpoint Rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \Delta x\left[f\left(\overline{x}_1\right)+f\left(\overline{x}_2\right)+\cdots+f\left(\overline{x}_n\right)\right]$$

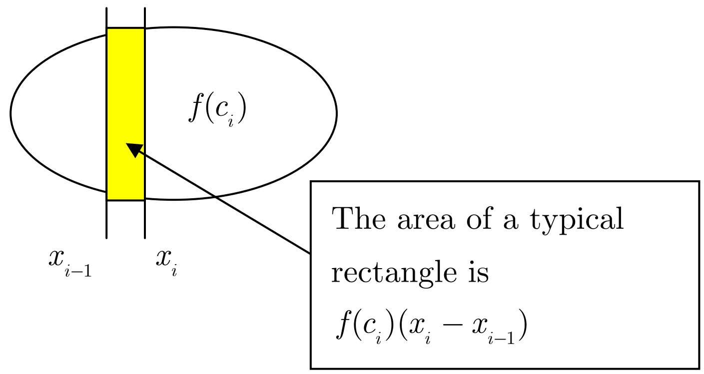
where $\Delta x=\dfrac{b-a}{n}$ and $\overline{x}_i=\dfrac{x_{i-1}+x_i}{2}$ = midpoint of $\left[x_{i-1},x_i\right]$.

Example 6: Using the midpoint rule

Use the midpoint rule to find rule with $n=6$ to approximate the integral $\int_0^1 e^x dx$.

Interpretations of the Definite Integral

1) **Area of a plane region**: Area of $S = \int_a^b f(x)dx$ where $f(x)$ is the length of a cross section of S.



2) **Mass of a string**: Total Mass $= \int_a^b f(x)dx$, where $f(x)$ is the density of the string at the point x.

3) **Distance traveled**: Total Distance $= \int_a^b f(t)dt$ where $f(t)$ is the velocity at time t.

4) **The volume of a solid region**: Volume of $S = \int_a^b A(x)dx$, where $A(x)$ is the cross-sectional area at x.

5) **Work**: If an object moves along a straight line by a force $f(x)$ that varies continuously, then the work, W, done in moving the object from $x = a$ to $x = b$ is $W = \int_a^b f(x)dx$

Example 6: Definite Integral as an Area

Find the area above the x-axis and under the function $f(x) = 2x + 5$ over the interval $[0,1]$.

The area is given by $\int_0^1 2x + 5dx$ from the previous examples we know that $\int_0^1 2x + 5dx = 6$.

Properties of the Antiderivative and the Definite Integral

Antiderivatives

Definition: An antiderivative of the function f is denoted $\int f(x)dx$. An **antiderivative** of the function f is any function F for which $F' = f$.

Properties of the antiderivative:

1. **Theorem 1**: If F and G are two antiderivatives of f on an interval $[a, b]$, then there is a constant c, such that $F(x) = G(x) + c$
2. **Theorem 2**: If f and g are two functions with antiderivatives $\int f(x)dx$ and $\int g(x)dx$, then the following hold:

    a) $\int cf(x)dx = c\int f(x)dx$ for any constant c.
    b) $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$.
    c) $\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$.

Notation: We agree to write $F(b) - F(a)$ as $F(x)\Big|_a^b$.

Terminology: In the definite integral $\int_a^b f(x)dx$ and in the indefinite integral $\int f(x)dx$, $f(x)$ is called the **integrand**.