Section 2: The Definite Integral

In this section, we investigate what happens to the values of an approximating sum when we let the number of intervals \( n \) in the partition go to infinity.

**Definition:** The mesh of a partition is the length of the longest section (or sections) in the partition. That is, the mesh of any partition is the largest value of \( x_k - x_{k-1} = \Delta x_k \) over all possible values of \( k \).

The Definite Integral

**Definition:** If \( f \) is a function defined on an interval \([a, b]\) and the sums
\[
\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})
\]
approach a certain number as the mesh of the partitions of \([a, b]\) shrink towards 0 (regardless of the choice of \( c_k \) in each interval \([x_{k-1}, x_k]\) ), that number is called the **definite integral** of \( f \) over \([a, b]\).

\[
\int_{a}^{b} f(x)dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta x_k
\]

The sum on the right-hand side is called a Reimann sum.

**Theorem: Existence of the definite integral.** Let \( f \) be a continuous function over an interval \([a, b]\). Then the approximating sums
\[
\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})
\]
approach a single number as the mesh of the partitions of \([a, b]\) approaches 0. Hence \( \int_{a}^{b} f(x)dx \) exists.
Example 1: Writing Reimann sums as integrals

Write the limit $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{1 + (x_k)^2} \Delta x_k$ as a definite integral on the interval $[0, \frac{\pi}{2}]$.

Example 2: Evaluating a definite integral

Find the definite integral of the function $f(x) = 2x + 5$ over the interval $[0,1]$. 
Example 3: **Evaluating a definite integral**

Use right-hand endpoints to express \( \int_{0}^{2} (2 - x^2) \, dx \) as a Reimann sum and then evaluate.

Example 4: **Writing an integral as a Reimann sum**

Express the integral \( \int_{0}^{\pi} \sin x \, dx \) as a Reimann sum. Use a calculator to approximate the sum of the first 50 terms.
Example 5: **Evaluating integrals by interpreting as areas**

Evaluate $\int_{-2}^{2} \sqrt{4-x^2} \, dx$ by interpreting in terms of an area.

The Midpoint Rule

**Theorem: Midpoint Rule**

\[
\int_a^b f(x) \, dx \approx \sum_{i=1}^{n} f(x_i) \Delta x = \Delta x \left[ f(x_i) + f(x_{i+1}) + \cdots + f(x_n) \right]
\]

where $\Delta x = \frac{b-a}{n}$ and $x_i = \frac{x_{i-1} + x_i}{2}$ = midpoint of $[x_{i-1}, x_i]$.

Example 6: **Using the midpoint rule**

Use the midpoint rule to find rule with $n = 6$ to approximate the integral $\int_0^1 e^x \, dx$.
Interpretations of the Definite Integral

1) **Area of a plane region**: Area of $S = \int_a^b f(x)dx$ where $f(x)$ is the length of a cross section of $S$.

![Diagram](https://via.placeholder.com/150)

2) **Mass of a string**: Total Mass $= \int_a^b f(x)dx$, where $f(x)$ is the density of the string at the point $x$.

3) **Distance traveled**: Total Distance $= \int_a^b f(t)dt$ where $f(t)$ is the velocity at time $t$.

4) **The volume of a solid region**: Volume of $S = \int_a^b A(x)dx$, where $A(x)$ is the cross-sectional area at $x$.

5) **Work**: If an object moves along a straight line by a force $f(x)$ that varies continuously, then the work, $W$, done in moving the object from $x = a$ to $x = b$ is $W = \int_a^b f(x)dx$

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**Example 6: Definite Integral as an Area**

Find the area above the $x$-axis and under the function $f(x) = 2x + 5$ over the interval $[0, 1]$.

The area is given by $\int_0^1 2x + 5dx$ from the previous examples we know that $\int_0^1 2x + 5dx = 6$. 

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Properties of the Antiderivative and the Definite Integral

Antiderivatives

**Definition:** An antiderivative of the function \( f \) is denoted \( \int f(x)dx \). An antiderivative of the function \( f \) is any function \( F \) for which \( F' = f \).

Properties of the antiderivative:

1. **Theorem 1:** If \( F \) and \( G \) are two antiderivatives of \( f \) on an interval \([a, b]\), then there is a constant \( c \), such that \( F(x) = G(x) + c \)

2. **Theorem 2:** If \( f \) and \( g \) are two functions with antiderivatives \( \int f(x)dx \) and \( \int g(x)dx \), then the following hold:

   a) \( \int cf(x)dx = c \int f(x)dx \) for any constant \( c \).
   b) \( \int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx \).
   c) \( \int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx \).

**Notation:** We agree to write \( F(b) - F(a) \) as \( F(x)|_a^b \).

**Terminology:** In the definite integral \( \int_a^b f(x)dx \) and in the indefinite integral \( \int f(x)dx \), \( f(x) \) is called the **integrand**.