Section 3: The Fundamental Theorems of Calculus

Properties of the definite integral:

1. If $b < a$, then $\int_a^b f(x)dx = -\int_b^a f(x)dx$.
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
4. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
5. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
6. If $f(x) \geq 0$ for all $x \in [a,b]$, $a < b$, then $\int_a^b f(x)dx \geq 0$.
7. If $f(x) \geq g(x)$ for all $x \in [a,b]$, $a < b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
8. If a, b, and c are numbers, then $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$.
9. If m and M are numbers and $m \leq f(x) \leq M$ for all $x \in [a,b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a), \text{ if } a < b, \text{ or}$$

$$m(b-a) \geq \int_a^b f(x)dx \geq M(b-a), \text{ if } b < a.$$

Example 1: Using the Properties of Definite Integrals

Suppose $\int_0^3 f(x)dx = 4$, $\int_3^6 f(x)dx = 4$ and $\int_2^6 f(x)dx = 5$. Find the following:

a) $\int_6^3 f(x)dx$

b) $\int_3^3 f(x)dx$

c) $\int_0^2 f(x)dx$

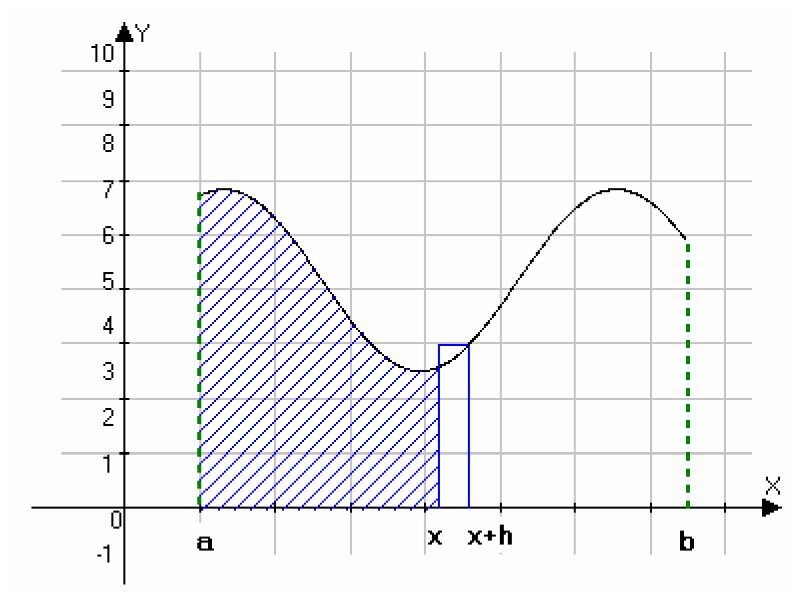
d) $\int_3^6 [3f(x)-2]\,dx$

e) $\int_0^6 [2f(x)-5]\,dx$

f) If $\int_0^3 f(x)dx = 12$ and $\int_0^6 f(x)dx = 42$, find the value of $\int_3^6 [2f(x)-3]dx$.

The Two Fundamental Theorems of Calculus

Theorem: **The First Fundamental Theorem of Calculus**. Let f be a continuous function defined on an open interval containing the interval $[a,b]$. Define $F(x)=\int_a^x f(t)dt$ for $a \le x \le b$. Then F is a differentiable function on $[a,b]$ and its derivative is f, that is $F'(x)=f(x)$.



Proof: Let f be continuous and define $F(x)=\int_a^x f(t)dt$ for $a \le x \le b$. Then, the derivative of $F(x)=\int_a^x f(t)dt$ is given by

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f-\int_a^{x} f\right)=\lim_{h\to 0}\frac{1}{h}\left(\int_a^{x} f+\int_x^{x+h} f-\int_a^{x} f\right)$$ By property 8

$$=\lim_{h\to 0}\frac{1}{h}\left(\int_x^{x+h} f\right)$$

Since f is continuous on the interval $[x, x+h]$, it achieves both a minimum value (m) and a maximum value (M) on this interval. Let $m = f(c)$ and $M = f(t)$ where $c, t \in [x, x+h]$.

Then by Property 9, we have

$$m(x+h-x) \le \int_x^{x+h} f \le M(x+h-x)$$

or

$$f(c)(h) \le \left(\int_x^{x+h} f\right) \le f(t)(h)$$

Therefore,

$$f(c) \le \frac{1}{h}\left(\int_x^{x+h} f\right) \le f(t)$$

Since $c, t \in [x, x+h]$, we must have (by continuity) $\lim_{h\to 0} f(t) = f(x) = \lim_{h\to 0} f(c)$.

By the Squeeze Theorem, $\frac{1}{h}\left(\int_x^{x+h} f\right) = f(x)$. If $h < 0$, a similar argument works.

Therefore,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\int_x^{x+h} f\right) = f(x).$$

Theorem: **The Second Fundamental Theorem of Calculus**. If f is a continuous function defined on an interval $[a,b]$ and if F is an antiderivative of f (so that $F' = f$), then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof: Since F and $\int_a^x f(t)dt$ have the same derivative, there is a constant C such that $\int_a^x f(t)dt = F(x) + C$. To find the constant we let $x = a$ to get

$0 = \int_a^a f = F(a) + C$. Therefore, $C = -F(a)$, and $\int_a^b f(x)dx = F(b) - F(a)$.

Corollary: Suppose f is a continuous function defined on an interval $[a,b]$. Then f is the derivative of some function.

Proof: Let $F(x)=\int_a^x f(t)dt$, then $F'(x)=f(x)$, that is $\frac{d}{dx}\left(\int_a^x f(t)dt\right)=f(x)$.

Example 2: Evaluating Definite Integrals

Evaluate $\int_1^2 (x+5)\,dx$

Solution: The integrand is $f(x)=x+5$. This has the general antiderivative $F(x)=\frac{x^2}{2}+5x+C$. We choose $C=0$, then

$$\int_1^2 (x+5)\,dx=\left.\frac{x^2}{2}+5x\right|_1^2=\frac{2^2}{2}+5\cdot 2-\left(\frac{1^2}{2}+5\cdot 1\right)=12-\frac{11}{2}=\frac{13}{2}$$

Example 3: Evaluating Definite Integrals

Evaluate $\int_0^4 \left(1+3x-x^2\right)dx$

Example 4: Evaluating Definite Integrals

Evaluate $\int_1^{\sqrt{2}} \frac{1}{x}\,dx$

Example 5: Evaluating Definite Integrals

Evaluate $\int_{\pi}^{2\pi} \cos\theta d\theta$

Example 6: Evaluating Definite Integrals

Evaluate $\int_{0}^{\pi} \sin\theta d\theta$

Example 7: Evaluating Definite Integrals

Evaluate $\int_{1}^{4} \frac{1}{\sqrt{x}} dx$

Example 8: Evaluating Definite Integrals

Evaluate $\int_0^2 \left(x^3 - 1\right)^2 dx$

Example 9: Evaluating Definite Integrals

Evaluate $\int_{\ln 3}^{\ln 6} 5e^x dx$

Example 10: Evaluating Definite Integrals

Evaluate $\int_0^{\pi/4} 5 \sec\theta \tan\theta d\theta$

Example 11: Evaluating Definite Integrals

Evaluate $\int_{-\pi/4}^{\pi/4} 3\sec^2\theta\, d\theta$

Example 12: Evaluating Definite Integrals

Evaluate $\int_{2}^{3} \ln 3 \cdot 3^x\, dx$

Example 13: Evaluating Definite Integrals

Evaluate $\int_0^1 \frac{dx}{1+x^2}$

Example 14: Evaluating Definite Integrals

Evaluate

$$\int_0^{0.5} \frac{dx}{\sqrt{1-x^2}}$$

Example 15: Using the Second Fundamental Theorem

Find g' if $g(x) = \int_1^x \ln t dt$

Example 16: Using the Second Fundamental Theorem

Find h' if $h(x) = \int_x^{10} \tan\theta d\theta$

Example 17: Using the Second Fundamental Theorem

Find g' if $g(x) = \int_0^{x^2} \sqrt{1+y^2} dy$