

Integration Using Substitution

Suppose $F(x)$ is any antiderivative of f , then from the chain rule, we have

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x) = f(g(x))g'(x)$$

It follows that

$$\int f(g(x))g'(x)dx = \int \frac{d}{dx}[F(g(x))]dx = F(g(x)) + c$$

If we cannot compute an integral such as $\int h(x)dx$ directly, we look for a new variable u and function $f(u)$ for which

$$\int h(x)dx = \int f(u(x))\frac{du}{dx}dx = \int f(u)du.$$

Obviously, we must choose the variable and function so that the second integral is easier than the first.

This technique involves using the chain rule in reverse. There are three steps in the process:

- 1) Choose a function $u(x)$ such that when both u and $du = u'(x)dx$ are substituted into the integral $\int h(x)dx$ we get a new integral of the form $\int f(u)du$. A common choice for $u(x)$ is the innermost expression of a composition of functions, e.g. the exponent or denominator. Look for terms that are derivatives of other terms.
- 2) Find a function $F(u)$ such that $F'(u) = f(u)$.
- 3) Eliminate u by substituting. The resulting function $H(x) = F(u(x))$ is an antiderivative of h ; so is any function of the form $H(x) + C$.

Example 1: Integration by Substitution

Evaluate $\int e^{3x} dx$

Solution:

Choose $u(x) = 3x$, then $du = u'(x)dx = 3dx$. Since the original integral contains dx , we solve to get $\frac{1}{3}du = dx$.

Now we replace $3x$ by u and dx by $\frac{1}{3}du$.

$$\begin{aligned}\int e^{3x} dx &= \int \frac{1}{3}e^u du \\ &= \frac{1}{3} \int e^u du \\ &= \frac{1}{3} e^u + C \\ &= \frac{1}{3} e^{3x} + C\end{aligned}$$

We can check by taking the derivative.

Example 2: Integration by Substitution

Evaluate $\int \frac{2x}{1+x^4} dx$

Solution:

Let $u = x^2$, then $du = 2xdx$. When we make this substitution we get:

$$\int \frac{2x}{1+x^4} dx = \int \frac{1}{1+u^2} du = \tan^{-1} u + c$$

Next, we replace u with x^2 in the final expression:

$$= \tan^{-1}(x^2) + c$$

Therefore, $\int \frac{2x}{1+x^4} dx = \tan^{-1}(x^2) + c$

Example 3: Integration by Substitution

Evaluate $\int 4 \sin(4x) dx$

Example 4: Integration by Substitution

Evaluate $\int 2xe^{x^2} dx$

Example 5: Integration by Substitution

Evaluate $\int x^2(x^3 + 1)^2 dx$

Example 6: Integration by Substitution

Evaluate $\int \frac{\tan^{-1} x}{1+x^2} dx$

Example 7: Integration by Substitution

Evaluate $\int \tan x dx$

Example 8: Integration by Substitution

Evaluate $\int \frac{e^{1/x}}{x^2} dx$

Example 9: Integration by Substitution

Evaluate $\int \frac{1+x}{1+x^2} dx$

Example 10: Integration by Substitution

Evaluate $\int \sqrt{3x-1} dx$

Example 11: Integration by Substitution

Evaluate $\int x\sqrt{3x-1} dx$

Example 12: Integration by Substitution

Evaluate $\int \sin^2(4x) \cos(4x) dx$

Example 13: Definite Integrals Involving Substitution

Evaluate $\int_0^1 x(x^2 + 3)^2 dx$

Solution: Let $u = x^2 + 3$, then $du = 2x dx$ or $x dx = \frac{1}{2} du$. A definite integral has limits of integration. So when we change from x to u we also have to change the limits. The equation $u = x^2 + 3$ expresses the relationship between x and u . When $x = 0$, $u = 0^2 + 3 = 3$ and when $x = 1$, $u = 1^2 + 3 = 4$. The original integral goes from 0 to 1, after substituting u the integral will go from 3 to 4:

$$\int_0^1 x(x^2 + 3)^2 dx = \int_0^1 \underbrace{(x^2 + 3)^2}_{u^2} \underbrace{x dx}_{\frac{1}{2} du} = \int_3^4 \frac{1}{2} u^2 du$$

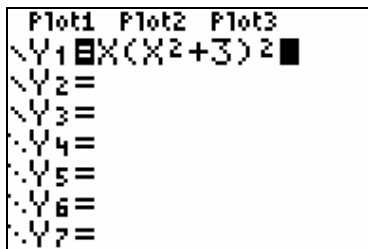
Evaluating, we have

$$\int_3^4 \frac{1}{2} u^2 du = \frac{1}{2} \int_3^4 u^2 du = \frac{1}{2} \frac{u^3}{3} \Big|_3^4 = \frac{u^3}{6} \Big|_3^4 = \frac{4^3}{6} - \frac{3^3}{6} = \frac{37}{6}$$

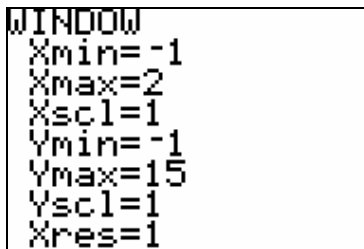
Therefore, $\int_0^1 x(x^2 + 3)^2 dx = \frac{37}{6}$

Check:

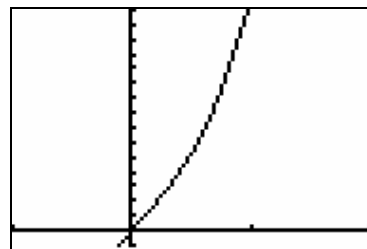
Input Function



Set the Window

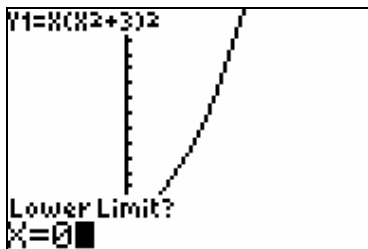


Graph the function

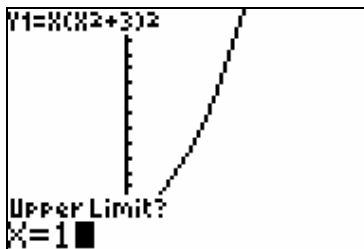


Select: Calc 7: $\int f(x) dx$

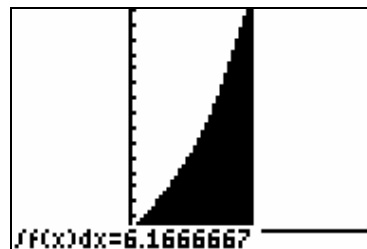
Set the lower limit:



Set the upper limit:



Select:



The calculator answer 6.166666667 is approximately equal to our answer of $\frac{37}{6}$.

Example 14: Definite Integrals Involving Substitution

Evaluate $\int_2^3 \frac{x}{\sqrt{7x-5}} dx$

Example 15: Definite Integrals Involving Substitution

Evaluate $\int_{-2}^6 x^2 \sqrt[3]{x+2} dx$

Example 16: Definite Integrals Involving Substitution

Evaluate $\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x dx$