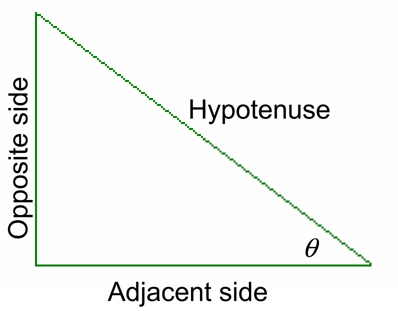
Trig Functions
Trig functions are not algebraic. Calculus uses *radian* measure unless otherwise stated.

There are three basic trig functions (sine, cosine and tangent) and three secondary trig functions (cotangent, secant and cosecant). Each function may be defined by using a right triangle, points in the plane, or by using points on the unit circle.

Trig Functions Defined using a Right Triangle



$\sin\theta = \frac{opp}{hyp}$ $\qquad$ $\csc\theta = \frac{hyp}{opp}$

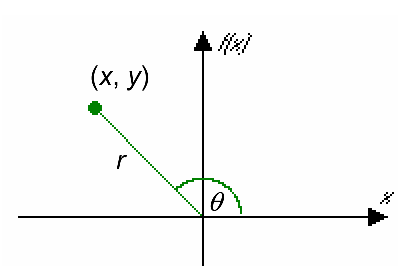
$\cos\theta = \frac{adj}{hyp}$ $\qquad$ $\sec\theta = \frac{hyp}{adj}$

$\tan\theta = \frac{opp}{adj}$ $\qquad$ $\cot\theta = \frac{adj}{opp}$

Knowledge of the values of the three basic trig functions for the special angles 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ is very helpful.

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
tan θ	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	0	∞

Trig Functions Defined using Points in the Plane

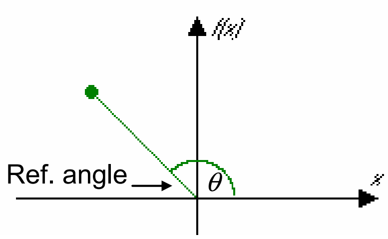


$\sin\theta = \frac{y}{r}$ $\qquad$ $\csc\theta = \frac{r}{y}$

$\cos\theta = \frac{x}{r}$ $\qquad$ $\sec\theta = \frac{r}{x}$

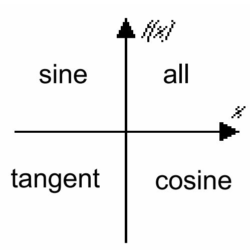
$\tan\theta = \frac{y}{x}$ $\qquad$ $\cot\theta = \frac{x}{y}$

Reference Angles
A reference angle for a given angle in the plane is measured from the terminal side of the angle to the *x*-axis. Reference angles are always positive. For example, the reference angle for $\theta = \frac{3\pi}{4}$ is $\frac{\pi}{4}$ and the reference angle for $\theta = 7\pi/6$ is $\pi/6$.



The absolute value of a trig function of a given angle is the same as the trig functions for the reference. The trig functions of a given angle may differ from the trig functions of its reference angle by the *sign*.

Quadrants and Trig Values
It is very helpful to know in which quadrants the trig functions are positive.

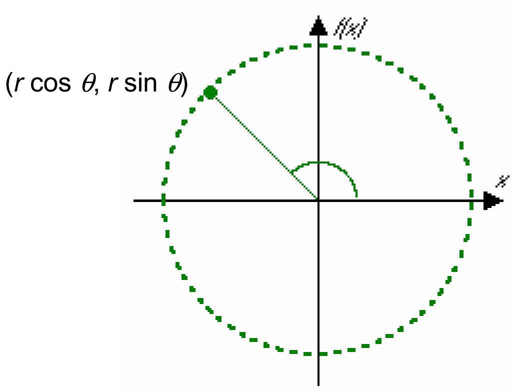


All trig function are positive in quadrant **I**.
Sine is positive in quadrant **II**.
Tangent is positive in quadrant **III**.
Cosine is positive in quadrant IV.

The phrase "**A**ll **S**tudent **T**ake **C**alculus" is often used to denote which trig functions are positive starting in quadrant I and moving counterclockwise.

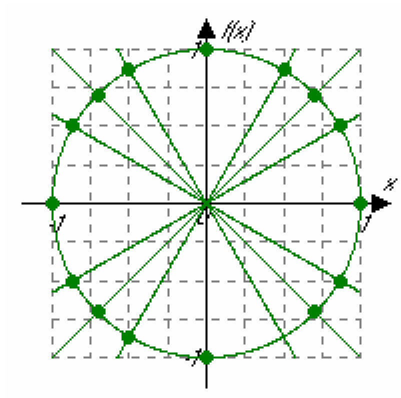
Using reference angles and the information about which quadrants contain positive trig function values, sin $\frac{3\pi}{4}$ = sin $\frac{\pi}{4}$ and sin $\frac{7\pi}{6}$ = - sin $\frac{\pi}{6}$.

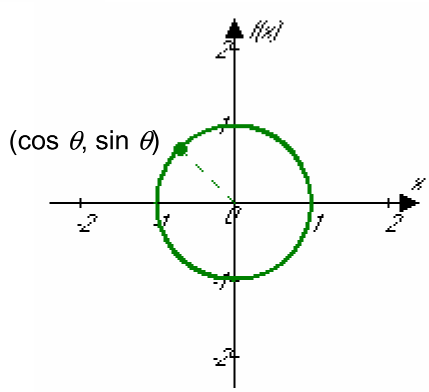
Points on a Circle
Points on the terminal side of an angle where it intersects a circle of radius *r* are ($r\cos\theta$, $r\sin\theta$).



Trig Functions Defined using Points on a Unit Circle

- The sine of an angle is defined to be the *y-* coordinate of the point where the terminal side of the angle touches the *unit* circle.
- The cosine of an angle is defined to be the *x-* coordinate of the point where the terminal side of the angle touches the unit circle.

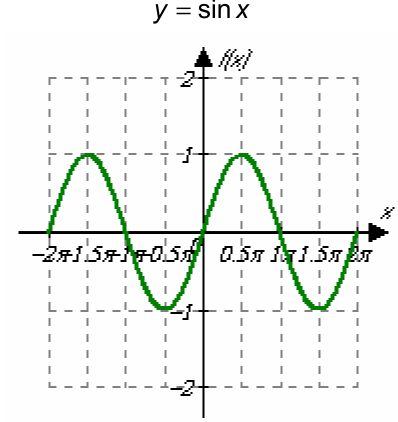




Basic Trig Functions

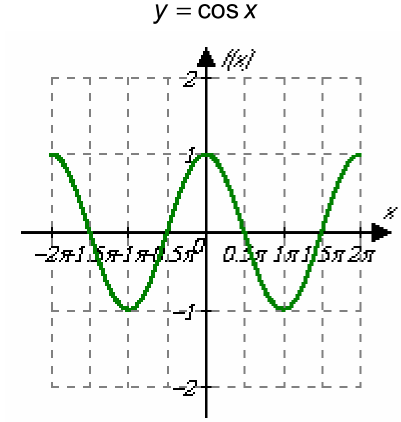
The basic trig functions are given below with their characteristics.

$y = \sin x$



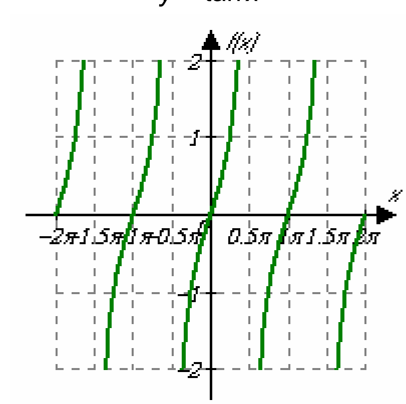
Amplitude = 1
Period = 2π (6.28)
Key Points:
(0, 0), ($\pi/2$, 1), (π, 0), ($3\pi/2$, -1)
Domain: $\Re$
Range: $-1 \le y \le 1$

$y = \cos x$



Amplitude = 1
Period = 2π (6.28)
Key Points:
(0, 1), ($\pi/2$, 0), (π -1), ($3\pi/2$, 0)
Domain: $\Re$
Range: $-1 \le y \le 1$

$y = \tan x$



Asymptotes: $x = (2k+1)\frac{\pi}{2}$

Period: = π
Key Points:
(0, 0), ($\pi/4$, 1), ($-\pi/4$, -1)

Domain: $x \neq (2k+1)\frac{\pi}{2}$

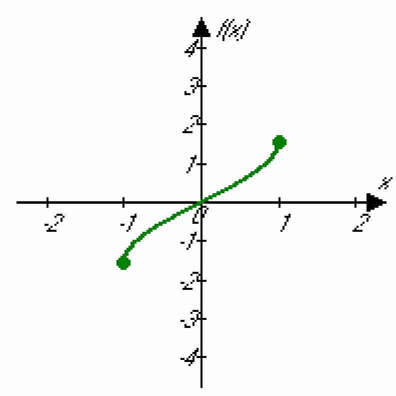
Range: $\Re$

Inverse Trig Functions

Inverse trig functions are functions that basically undo the trig function. In one context, each inverse trig function represents an angle whose trig function is given.

Because we want these inverses to be *functions*, we limit their outputs so that each input produced one output and each output comes from exactly one input.

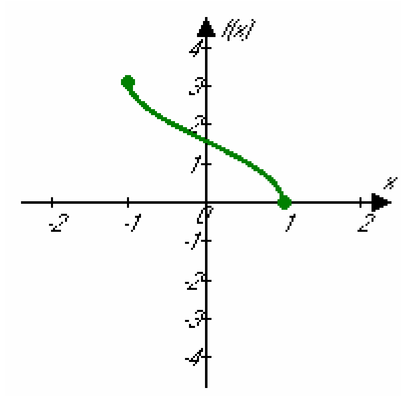
$y = \sin^{-1} x$



Key Points:
(0, 0), (1, π/2), (0, π,), (-1, 3π/ -1)
Domain: $-1 \le x \le 1$

Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

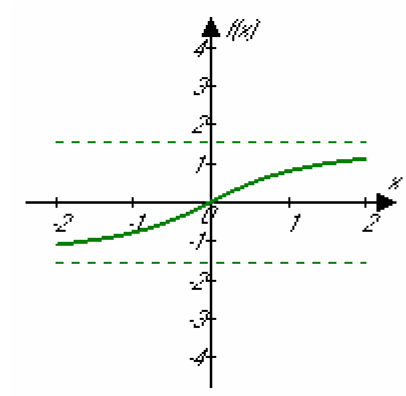
$y = \cos^{-1} x$



Key Points:
(1, 0), (0, π/2,), (-1, π), (0, 3π/2,)
Domain: $-1 \le x \le 1$

Range: $0 \le y \le \pi$

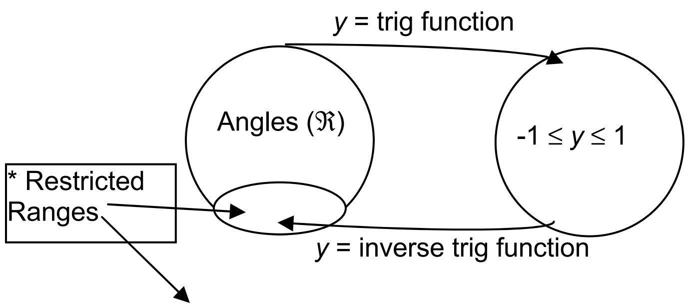
$y = \tan^{-1} x$



Key Points:
(0, 0), (π/4, 1), (-π/4, -1)
Domain: $\Re$

Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

The relationship between the sine function and its inverse function is illustrated below. The other trig functions behave in a similar manner.



$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$
quad I or quad IV

$0 \le \cos^{-1} x \le \pi$
quad I or quad II

$-\frac{\pi}{2} \le \tan^{-1} x \le \frac{\pi}{2}$
quad I or quad IV

Fundamental Trig Identities
Trig identities are used to simplify trig expressions. The fundamental trig identities are:

$\sin^2 x + \cos^2 x = 1$ $\quad$ $\tan^2 x + 1 = \sec^2 x$ $\quad$ $1 + \cot^2 x = \csc^2 x$

$\tan x = \frac{\sin x}{\cos x}$ $\quad$ $\sec x = \frac{1}{\cos x}$ $\quad$ $\csc x = \frac{1}{\sin x}$ $\quad$ $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

$\sin(-x) = -\sin x$ $\quad$ $\cos(-x) = \cos x$ $\quad$ $\tan(-x) = -\tan x$

$\sin\left(\frac{\pi}{2} - x\right) = \cos x$ $\quad$ $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ $\quad$ $\tan\left(\frac{\pi}{2} - x\right) = \cot x$

sin(*A* + *B*) = sin *A* cos *B* + sin *B* cos *A* $\quad$ sin(*A* – *B*) = sin *A* cos *B* – sin *B* cos *A*
cos(*A* + *B*) = cos *A* cos *B* – sin*A* sin *B* $\quad$ cos(*A* – *B*) = cos *A* cos *B* + sin *A* sin *B*

Solving Trig Equations
Most trig equations have two solutions within one complete revolution. To solve a trig equation:

1. Find one solution by using inverse trig functions.
2. Find a second solution by using information about which quadrants contain trig functions that have the same sign.
3. Add or subtract multiples of 2π to each of the solutions found above to get other solutions.

Example #1: Find all solutions to $\sin x = \frac{1}{2}$.

Solution #1 (For Special-Angle Equations):

1. $\sin\theta = \frac{1}{2}$ when $\theta = \frac{\pi}{6}$.
2. The sine function is also positive in quadrant II, so $\theta = \pi - \pi/6 = 5\pi/6$ is also a solution.
3. All angles whose terminal side coincides with $\pi/6$ and $5\pi/6$ are also solutions.

The solutions are $\pi/6 + 2k\pi$ and $5\pi/6 + 2k\pi$, where k is any integer .
These solutions, letting k = -1, 0, and 1, are

$$\{\cdots, \frac{5\pi}{6} - 4\pi, \frac{\pi}{6} - 2\pi, \frac{5\pi}{6} - 2\pi, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{6} + 2\pi, \frac{5\pi}{6} + 2\pi, \cdots\}$$

$$\{\cdots, -\frac{19\pi}{6} - \frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \cdots\}$$

$\approx$ {…, -9.95, -5.76, -3.67, 0.52, 2.62, 6.81, 8.90, …}

Solution #2 (For any Equation)

1. Get a single trig term equal to a number by using identities or algebra:

   $\sin x = \frac{1}{2}$ has the desired form.

2. Take the appropriate inverse trig function to both sides to set one solution.

   $$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{1}{2}\right)$$

   $x_1 \approx 0.52$

   (The value found from the calculator is in quadrant I)

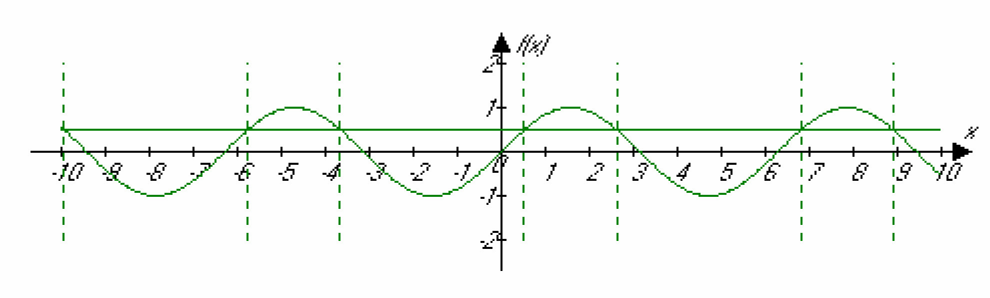
3. Determine the other solution within one complete revolution.
   Because the sine function is also positive in quadrant II,

   $x_2 \approx = 3.1419 - 0.5235 = 2.62$

4. Write the general solution and some specific solutions by adding 2π times some integer k.
   Other solutions include:

   {…, 0.52 – 6.28, 2.62 – 6.28, 0.52, 2.62, 0.52 + 6.28, 2.62 + 6.28, …}
   {…, -5.76, -3.66, 0.52, 2.62, 6.80, 8.90, …}

Reviewing the graph of its associated trig function may be used to check the solutions of a trig equation. Graph $y = \sin x$ and $y = ½$. The points of intersection of the two graphs are the solutions and are indicated by dashed vertical lines in the illustration below.



Exercise Identify each function and label the points of intersection of the two graph.

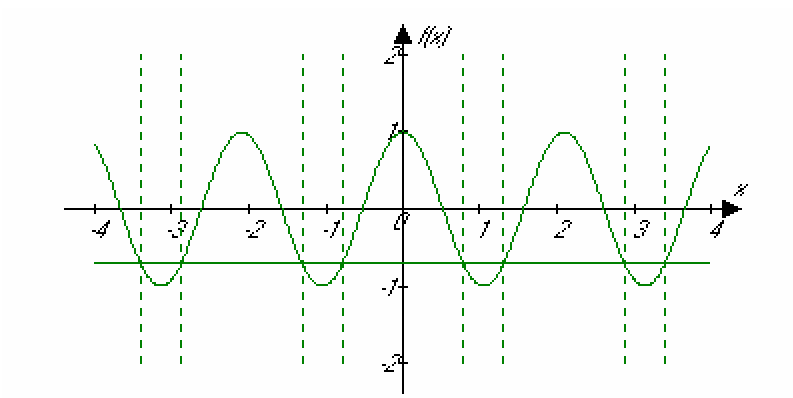
Example #2: Find all solutions to $2\cos 3x = -\sqrt{2}$.

1. Sove for cos 3*x*: cos 3*x* = $-\dfrac{\sqrt{2}}{2}$
2. $\cos\theta = -\dfrac{\sqrt{2}}{2}$ when $\theta = \dfrac{3\pi}{4}$, which is in quadrant II. The reference angle is $\dfrac{\pi}{4}$.
3. The cosine is also negative in quadrant III, so $\pi + \dfrac{\pi}{4} = \dfrac{5\pi}{4}$ is also a solution.
4. The general solutions of $\cos\theta = -\dfrac{\sqrt{2}}{2}$ have the form $\dfrac{3\pi}{4} + 2k\pi$ and $\dfrac{5\pi}{4} + 2k\pi$.
5. $\theta = 3x$, so $3x = \dfrac{3\pi}{4} + 2k\pi$ or $3x = \dfrac{5\pi}{6} + 2k\pi$.
6. Solving each of these equation yields:

$$3x = \frac{3\pi}{4} + 2k\pi \quad \text{or} \quad 3x = \frac{5\pi}{4} + 2k\pi$$

$$x = \frac{\pi}{4} + \frac{2k\pi}{3} \quad \text{or} \quad x = \frac{5\pi}{12} + \frac{2k\pi}{3}$$

Combining the solutions: x ≈ …, -3.40, -2.88, -1.31, -0.79, 0.79, 1.31, 2.88, 3.40, …



Exercise: Identify each graph and label the points of intersection.

Exercise: Find all solutions of $4\sin\theta\cos\theta = \sqrt{3}$.

Sinusoidal Functions
Any function that can be written in the following form is sinusoidal.

$$f(x) = a \sin b(x - c) + d$$

The specific values of *a*, *b*, *c*, and *d* determine the orientation and placement of the graph.

a is the amplitude of the function If *a* is negative, the graph is reflected vertically.

b is related to the period of the function: period = $\frac{2\pi}{b}$

c is the phase shift of the function
d is the vertical shift of the function and is the midline

$y = -3\sin\left(\frac{1}{2}(x-\pi)\right) - 1$ is shown below.