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Calc II Integration Review Spring 2006

Please show all work neatly on this paper.

1. $\int (x^2 + 5x)dx$ $\quad = \int x^2 dx + \int 5x dx$

$$= \frac{x^3}{3} + \frac{5}{2}x^2 + C$$

2. $\int (\frac{1}{x^2} + 5\sin x)dx$ $\quad = \int x^{-2} + 5\sin x dx$

$$= -x + (-5\cos x) + C$$

$$= -5\cos x - \frac{1}{x}$$

3. $\int (e^x + 5x^{3/5})dx$ $\quad = e^x + \frac{25}{8}x^{\frac{8}{5}} + C$

4. $\int \frac{1}{2}(e^x + e^{-x})dx$ $\quad = \frac{1}{2}\left(e^x - e^{-x}\right) + C$ This is sinh function

BTW, This is the cosh function

5. $\int (5 + \frac{1}{\sqrt[3]{x}})dx$ $\quad = 5x + \frac{3}{2}x^{\frac{2}{3}} + C$

6. $\int (\cos x + 5\sec^2 x)dx$ $= \sin x + 5\tan x + C$

7. $\int (\sec x \tan x)dx$ $= \sec x + C$

8. $\int (\csc x \cot x)dx$ $= -\csc x + C$

9. $\int \frac{x^2 + 5}{x} dx$

$$= \int \frac{x^2}{x} + \frac{5}{x} dx$$

$$= \int x + \frac{5}{x} dx$$

$$= \frac{1}{2}x^2 + 5\ln|x| + C$$

10. $\int e^x(2 + e^{-x})dx$

$$= \int 2e^x + e^0 dx$$

$$= 2e^x + x + C$$

11. $\int x(x^2 + 5x)dx$

$$= \int x^3 + 5x^2 dx$$

$$= \frac{x^4}{4} + \frac{5x^3}{3} + C$$

12. $\int \frac{2x}{x^2 + 7} dx$

$$= \int \frac{1}{u} du$$

$$= \ln|u| + C$$

let u $= x^2 + 7$

then du $= 2xdx$

$$= \ln\left(x^2 + 7\right) + C$$

13. $\int \frac{1}{\sqrt{1 - x^2}} dx$

This integral is easily recognizable as…

$\sin^{-1} x + C$

14. $\int \cot x dx = \int \frac{\cos x}{\sin x} dx$

$$= \int \frac{1}{u} du$$

Let $u = \sin x$

then, $du = \cos x dx$

$$= \ln|u| + C$$

$$= \ln \sin(x) + C$$

(Not sure whether ABS value notation is needed?)

15. $\int \tan x dx = \int \frac{\sin x}{\cos x} dx$

$$Let\ u = \cos x$$
$$then,\ du = -\sin x dx$$
$$-du = \sin x dx$$

$$= -\int \frac{1}{u} du$$
$$= -\ln|u| + C$$
$$= -\ln|\cos x| + C$$
$$= \ln|\sec x| + C$$

16. $\int \frac{\ln x}{x} dx$

$$Let\ u = \ln x$$
$$Then,\ du = \frac{1}{x} dx$$

$$= \int u du$$
$$= \frac{u^2}{2} + C$$
$$= \frac{(\ln x)^2}{2} + C$$

17. $\int \frac{1}{x \ln x} dx$

$$Let\ u = \ln x$$
$$Then,\ du = \frac{1}{x} dx$$

$$= \int \frac{1}{u} du$$
$$= \ln|u| + C$$
$$= \ln \ln|x| + C$$

18. $\int \frac{x}{\left(x^2+7\right)^4}dx$

Let $u = x^2 + 7$

Then, $du = 2xdx$

$\frac{1}{2}du = xdx$

$$\frac{1}{2}\int \frac{1}{u^4}du$$

$$= \frac{1}{2}\left(\frac{u^{-3}}{-3}\right) + C$$

$$= -\frac{1}{6}u^{-3} + C$$

$$= -\frac{1}{6}\left(x^2+7\right)^{-3} + C$$

or, if you like...$-\frac{1}{6\left(x^2+7\right)^3} + C$

19. $\int \frac{\sin x dx}{\cos^2 x}$

Let $u = \cos x$

Then, $du = -\sin xdx$

$-du = \sin xdx$

$$= -\int \frac{1}{u^2}du$$

$$= -\int u^{-2}du$$

$$= -\frac{u^{-1}}{-1} + C$$

$$= \frac{1}{u} + C$$

$$= \frac{1}{\cos x} + C$$

Just for fun let's check this one: $\frac{d}{dx}\left(\sec x + c\right)$

$$= \sec x \tan x$$

$$= \frac{\sin x}{\cos^2 x}$$

20. $\int e^x \cos e^x dx$

$Let\ u = e^x$

$Then,\ du = e^x du$

$= \int \cos(u)\, du$

$= \sin(u) + C$

$= \sin(e^x) + C$

21. $\int \frac{dx}{\sqrt{3x+7}}$

$Let\ u = 3x + 7$

$Then,\ du = 3dx$

$= \frac{1}{3}\int u^{\frac{-1}{2}} du$

$= \frac{1}{3}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right] + C$

$= \frac{2}{3} u^{\frac{1}{2}} + C$

$= \frac{2}{3}(3x+7)^{\frac{1}{2}} + C$

22. $\int \frac{dx}{(3x+5)^4}$

$Let\ u = 3x + 5$

$Then,\ du = 3dx$

$\frac{1}{3} du = dx$

$= \frac{1}{3}\int (u)^{-4}\, du$

$\frac{1}{3}\left[\frac{u^{-3}}{-3}\right] + C$

$= -\frac{1}{9}(3x+5)^{-3} + C$

23. $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^4}$

$Let\ u = 1 - \sqrt{x}$

$Then,\ du = \frac{1}{2\sqrt{x}}$

$2du = \frac{1}{\sqrt{x}}$

$= 2\int u^{-4} du$

$= 2\left[\frac{u^{-3}}{-3}\right] + C$

$= -\frac{2}{3}u^{-3} + C$

$= -\frac{2}{3\left(1+\sqrt{x}\right)^3} + C$

24. $\int \frac{\cos(\ln x)\,dx}{x}$

$Let\ u = \ln x$

$Then,\ du = \frac{1}{x}dx$

$= \int \cos u\,du$

$= \sin u + C$

$= \sin(\ln x) + C$

25. $\int \frac{\ln x^2}{x} dx$

$Let\ u = \ln x^2$

$Then,\ du = \frac{2}{x}dx$

$\frac{1}{2}du = \frac{1}{x}dx$

$= \frac{1}{2}\int u\,du$

$= \frac{1}{2}\left[\frac{1}{2}u^2\right] + C$

$= \frac{1}{4}\left(\ln x^2\right)^2 + C$

26. $\int \frac{xdx}{(ax+b)^2}, a \neq 0$

Let $u = ax + b$

Then, $du = adx$

$\frac{1}{a}du = dx$

Still we have x so,

$u - b = ax$

$\frac{u-b}{a} = x$

$$= \frac{1}{a}\int \frac{\frac{u-b}{a}}{a^2} du$$

$$= \frac{1}{a}\int \frac{u-b}{a} \times \frac{1}{u^2} du$$

$$= \frac{1}{a}\int \frac{u-b}{au^2} du$$

$$= \frac{1}{a}\int \left[\frac{u}{au^2} du - \int \frac{b}{au^2} du \right]$$

$$\frac{1}{a^2}\int \frac{1}{u} du - \frac{b}{a^2}\int u^{-2} du$$

$$\frac{1}{a^2}\ln|u| - \left(\frac{b}{a^2}\right)\left(-u^{-1}\right) + C$$

$$= \frac{1}{a^2}\left[\ln|u| + \frac{b}{u}\right] + C$$

$$= \frac{1}{a^2}\left[\ln(ax+b) + \frac{b}{ax+b}\right] + C$$

27. $\int \frac{x^2 dx}{(ax+b)^2}, a \neq 0$

$$= \frac{1}{a}\int \frac{du}{u^2}$$

$Let\ u = ax + b \rightarrow x = \frac{u-b}{a}$ So, we have $x^2 = \frac{(u-b)^2}{a^2}$

$Then,\ du = adx$

$\frac{1}{a}du = dx$

Integral continued here:

$$= \frac{1}{a^3}\int \left(\frac{u^2 - 2ub + b^2}{u^2} \right)$$

$$= \frac{1}{a^3}\int \left(1 - \frac{2b}{u} + b^2 u^{-2} \right)$$

$$= \frac{1}{a^3}\left[u - 2b\ln|u| - \frac{b^2}{u} \right] + C$$

$$= \frac{1}{a^3}\left[ax + b - 2b\ln|ax+b| - \frac{b^2}{ax+b} \right] + C$$

28. $\int_0^2 \left(x^2 - 2\right) dx$

$$= \frac{x^3}{3} - 2x \Big|_0^2$$

$$= \left[\frac{2^3}{3} - 2(2) \right] - \left[\frac{0^3}{3} - 2(0) \right]$$

$$= \frac{8}{3} - \frac{12}{3}$$

$$= -\frac{4}{3}$$

29. $\int_{-1}^{2} e^{3x}\,dx$

$Let\ u = 3x$

$Then,\ du = 3dx$

$\frac{1}{3}du = dx$

$= \frac{1}{3}\int_{-3}^{6} e^{u}\,du$

$= \frac{1}{3}\left[e^{6} - e^{-3}\right]$

$= \frac{1}{3}\left(e^{6} - \frac{1}{e^{3}}\right)$

30. $\int_{2}^{3} \frac{e^{1/x}}{x^{2}}\,dx$

$Let\ u = \frac{1}{x}$

$Then,\ du = -\frac{1}{x^{2}}dx$

$-du = \frac{1}{x^{2}}dx$

Since we've changed variables, we'll change our limits. We do this by evaluating u using our original limits.

$\frac{1}{x} \to \frac{1}{3}, \frac{1}{2}$

$= -\int_{\frac{1}{2}}^{\frac{1}{3}} e^{u}\,du$

$= -\left(e^{\frac{1}{3}} - e^{\frac{1}{2}}\right)$

$= -e^{\frac{1}{3}} + e^{\frac{1}{2}}$

31. $\int_{\pi/6}^{\pi/4} \tan x \sec^2 x dx$

$Let\ u = \tan x$

$Then,\ du = \sec^2 xdx$

Once again, let us change our limits:

$\tan x$

$\tan\left(\frac{\pi}{4}\right) = 1$

$\tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

$$= \int_{\frac{\sqrt{3}}{3}}^{1} udu$$

$$= \frac{u^2}{2}\Big|_{\frac{\sqrt{3}}{3}}^{1}$$

$$= \frac{1^2}{2} - \frac{\left(\frac{\sqrt{3}}{3}\right)^2}{2}$$

$$= \frac{1}{2} - \frac{1}{6}$$

$$= \frac{1}{3}$$

32. $\int_1^e \frac{(\ln x)^3}{x} dx$

$Let\ u = \ln x$

$Then,\ du = \frac{1}{x}dx$

$Limits:$

$\ln 1 = 0$

$\ln e = 1$

$$\int_0^1 u^3 du$$

$$= \frac{u^4}{4}\Big|_0^1$$

$$= \frac{1^4}{4} - \frac{0^4}{4}$$

$$= \frac{1}{4}$$

33. $\int_0^4 f(x)dx$ if $f(x) = \begin{cases} x-1 & x<1 \\ 2 & x \geq 1 \end{cases}$

$$= \int_0^4 (x-1)\,dx + \int_0^4 2dx$$

$$= \frac{x^2}{2} - x + 2x$$

34. Differentiate $f(x) = \int_1^x t^4 dt$

$$f'(x) = x^4$$

35. Differentiate $f(x) = \int_1^{\sqrt{x}} \frac{1}{t^4}\,dt$

$$f'(x) = \frac{1}{\sqrt{x^4}}$$

$$= \frac{1}{\sqrt{x^4}} \bullet \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x^5}}$$

36. Differentiate $f(x) = \int_0^{\sin x} \sqrt{1+t^3}\,dt$

$$f'(x) = \sqrt{1+(\sin x)^3}\,(\cos x)$$

37. Differentiate $f(x) = \int_{x^2}^{15} \sin^3 2t\,dt$

Need to flip the integral…

$$= -\int_{15}^{x^2} \sin^3 2t\,dt$$

$$f'(x) = \sin^3 2x^2 \bullet 2x$$

$$= -2x\sin^3 2x^2$$

38. Differentiate $f(x) = \int_{2x}^{3x} t\tan t\,dt$

$$f(x) = -\int_{2x}^{0} x\tan x\,dx + \int_{0}^{3x} x\tan x\,dx$$

$$f'(x) = -2(2x)\tan 2x + 3(3x)\tan 3x$$

$$= -4x\tan 2x + 9x\tan 3x$$

39. How often should a machine be overhauled? This depends on the rate $f(t)$ at which it depreciates and the cost *A* of overhaul. Denote the time interval between overhauls by *T*.

a) Explain why you would like to minimize $g(T) = \dfrac{\left[A + \int_0^T f(t)dt\right]}{T}$.

We want to minimize the overall cost of the machine overhauls.

b) Find $\dfrac{dg}{dT}$.

$$= \frac{T\left[f(t)\right] - \left[A + \int_0^T f(t)\,dt\right]}{T^2}$$

c) Show that when $\dfrac{dg}{dT} = 0$, $f(T) = g(T)$

$$T\left[f(t)\right] - \left[A + \int_0^T f(t)\,dt\right] = 0$$

$$f(t) = \frac{A + \int_0^T f(t)\,dt}{T} = g(t)$$

d) Is this reasonable?

Sure, why not?

Function	**Derivative**	**Integral**	**Function**
x^n	nx^{n-1}	$\int x^n$	$\frac{x^{n+1}}{n+1}+C, n \neq 1$
$\ln\lvert x\rvert$	$\frac{1}{x}$	$\int \frac{1}{x}$	$\ln\lvert x\rvert + C$
$\sin x$	$\cos x$	$\int \cos x$	$\sin x + C$
$\cos x$	$-\sin x$	$\int \sin x$	$-\cos x + C$
$\tan x$	$\sec^2 x$	$\int \sec^2 x$	$\tan x + C$
$\sec x$	$\sec x \tan x$	$\int \sec x \tan x$	$\sec x + C$
$\csc x$	$-\csc x \cot x$	$\int \csc x \cot x$	$-\csc x + C$
$\cot x$	$-\csc^2 x$	$\int \csc^2 x$	$-\cot x + C$
e^x	e^x	$\int e^x$	$e^x + C$
b^x	$\ln b \cdot b^x$	$\int b^x$	$\frac{1}{\ln b} b^x + C$
$\log_b x$	$\frac{1}{x \ln b}$		
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{1-x^2}}$	$\cos^{-1} x + C$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}$	$\tan^{-1} x + C$
k (constant)	0	$\int k$	$kx + C$
kf	kf'	$\int kf$	$k\int f$
$f \pm g$	$f' \pm g'$	$\int f \pm g$	$\int f \pm \int g$
f/g	$\frac{gf' - fg'}{g^2}$		
fg	$fg' + gf'$	$\int fg$	Integration by parts

$(f \circ g)(x)$	$f'(g(x))g'(x)$	$\int f'(g(x))g'(x)dx$	u-substitution: Let $g(x) = u$, then $= \int f'(u)du$

Antiderivatives

Definition: An **antiderivative** of the function f is any function F for which $F' = f$. An antiderivative of the function f is denoted $\int f(x)dx$.

Theorem: If F is an **antiderivative** of the function f on an interval I, then the most general antiderivative of f on I is $F(x) + C$ where C is any constant.

Properties of the antiderivative:

1. **Theorem 1**: If F and G are two antiderivatives of f on an interval $[a,b]$, then there is a constant c, such that $F(x) = G(x) + c$

2. **Theorem 2**: If f and g are two functions with antiderivatives $\int f(x)dx$ and $\int g(x)dx$, then the following hold:
   a) $\int cf(x)dx = c\int f(x)dx$ for any constant c.
   b) $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$.
   c) $\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$.

Application

If the acceleration of an object at any time is known and the initial values of the objects position, $s(0)$, and its initial velocity, $v(0)$, are known, then the objects position at any time can be found. This is true since:

If $s(t)$ is given then,	If $a(t)$ is given then,
$v(t) = s'(t)$	$v(t) = at + v(0)$
$a(t) = v'(t)$	$s(t) = a\frac{t^2}{2} + v(0)t + s(0)$

Notation: We agree to write $F(b)-F(a)$ as $F(x)\Big|_a^b$.

Terminology: In the definite integral $\int_a^b f(x)dx$ and in the indefinite integral $\int f(x)dx$, $f(x)$ is called the **integrand**.

Theorem: The First Fundamental Theorem of Calculus. Suppose f is a continuous function defined on an interval $[a,b]$ and also that F is an antiderivative of f (so $F' = f$), then $\int\limits_a^b f(x)dx = F(b)-F(a)$.

Theorem: The Second Fundamental Theorem of Calculus. Let f be a continuous function defined on an open interval containing the interval $[a,b]$. Define $G(x) = \int\limits_a^x f(t)dt$ for $a \le x \le b$. Then G is a differentiable function on $[a,b]$ and its derivative is f, that is $G'(x) = f(x)$.

Corollary: Suppose f is a continuous function defined on an interval $[a,b]$. Then f is the derivative of some function.

The Definite Integral

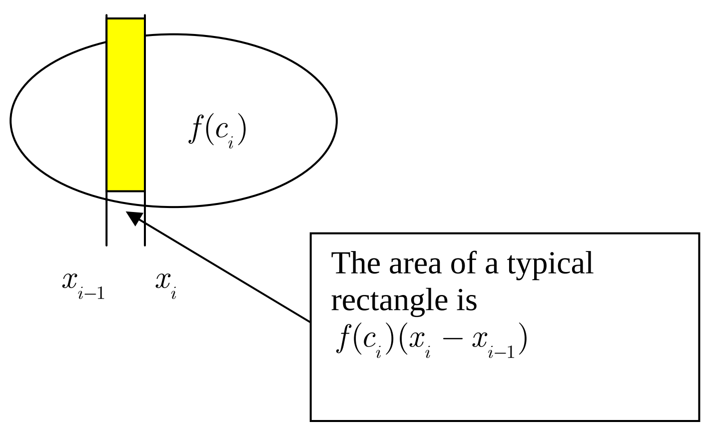
Definition: The definite integral of a function f over an interval $[a,b]$. If f is a function defined on an interval $[a,b]$ and the sums $\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$ approach a certain number as the length of all the intervals $[x_{k-1}, x_k]$ shrink towards 0 (regardless of the choice of c_k in each interval $[x_{k-1}, x_k]$), that number is called the **definite integral** of f over $[a,b]$.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

The sum on the right-hand side is called a **Reimann sum**.

Interpretations of the Definite Integral

1) **Area of a plane region**: Area of $S = \int_a^b f(x)dx$ where $f(x)$ is the length of a cross section of S



2) **Mass of a string**: Total Mass $= \int_a^b f(x)dx$, where $f(x)$ is the density of the string at the point x

3) **Distance traveled**: Total Distance $= \int_a^b f(t)dt$ where $f(t)$ is the velocity at time t

4) **The volume of a solid region**: Volume of $S = \int_a^b A(x)dx$, where $A(x)$ is the cross-sectional area at x

5) **Work**: If an object moves along a straight line by a force $f(x)$ that varies continuously, then the work, *W*, done in moving the object from $x = a$ to $x = b$ is $W = \int_a^b f(x)dx$