Section 1 Integration Using Substitution

Suppose $F(x)$ is any antiderivative of *f*, then from the chain rule, we have

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))g'(x) = f(g(x))g'(x)$$

It follows that

$$\int f(g(x))g'(x)dx = \int \frac{d}{dx}\left[F(g(x))\right]dx = F(g(x)) + c$$

If we cannot compute an integral such as $\int h(x)dx$ directly we look for a new variable u and function $f(u)$ for which

$$\int h(x)dx = \int f(u(x))\frac{du}{dx}dx = \int f(u)du\,.$$

Obviously, we must choose the variable and function so that the second integral is *easier* than the first.

This technique involves using the chain rule in reverse. There are three steps in the process:

1) Choose a function $u(x)$ such that when both u and $du = u'(x)dx$ are substituted into the integral $\int h(x)dx$ we get a new integral of the form $\int f(u)du$. A common choice for $u(x)$ is the innermost expression of a composition of functions, e.g. the exponent or denominator. Look for terms that are derivatives of other terms.

2) Find a function $F(u)$ such that $F'(u) = f(u)$.

3) Eliminate u by substituting. The resulting function $H(x) = F(u(x))$ is an antiderivative of *h*; so is any function of the form $H(x) + C$.

Example 1: Integration by Substitution

Find $\int e^{3x} dx$

Solution: Let $u(x) = 3x$, then $du = u'(x)dx = 3dx$. Since the original integral contains dx , we solve to get $\frac{1}{3}du = dx$.

Now we replace $3x$ by u and dx by $\frac{1}{3}du$.

$$\begin{aligned}\int e^{3x} dx &= \int \frac{1}{3} e^{u} du \\ &= \frac{1}{3}\int e^{u} du \\ &= \frac{1}{3}e^{u} + C \\ &= \frac{1}{3}e^{3x} + C\end{aligned}$$

We can check by taking the derivative.

Example 2: Integration by Substitution

Evaluate $\int \frac{2x}{1+x^4} dx$

Solution: Let $u = x^2$, then $du = 2xdx$. When we make this substitution we get:

$$\int \frac{2x}{1+x^4} dx$$

$$= \int \frac{1}{1+u^2} du$$

$$= \tan^{-1} u + c$$

Next, we replace u with x^2 in the final expression:

$$= \tan^{-1}\left(x^2\right) + c$$

Therefore, $\int \frac{2x}{1+x^4} dx = \tan^{-1}\left(x^2\right) + c$

Example 3: Integration by Substitution

Evaluate using substitution.

$$\int \sin(3x)dx$$

Example 4: Integration by Substitution

Evaluate using substitution.

$$\int x^3(x^4+1)^2dx$$

Example 5: Integration by Substitution

Evaluate using substitution.

$$\int \sqrt{2x-1}dx$$

Example 6: Integration by Substitution

Evaluate using substitution.

$$\int x\sqrt{2x-1}dx$$

Example 7: Integration by Substitution

Evaluate using substitution.

$$\int \sin^2(3x)\cos(3x)dx$$

Example 8: Evaluating Definite Integrals using Substitution

Evaluate $\int_0^2 x\left(x^2+1\right)dx$.

Solution: Since x is a multiple of the derivative of x^2+1, we let $u=x^2+1$, then $du=2xdx$. Since the integrand contains xdx, we solve for this: $\frac{1}{2}du=xdx$. With this choice of u, and using the fact that $u(0)=1$ and $u(2)=5$, we have

$$\begin{aligned}\int_0^2 x\left(x^2+1\right)dx &= \frac{1}{2}\int_1^5 u\,du \\ &= \frac{1}{2}\left(\left.\frac{u^2}{2}\right|_1^5\right)=\left.\frac{u^2}{4}\right|_1^5=\frac{5^2}{4}-\frac{1^2}{4}=\frac{25}{4}-\frac{1}{4}=6\end{aligned}$$

Example 9: Evaluating Definite Integrals using Substitution

Evaluate $\int_2^3 \frac{x}{\sqrt{7x-5}}dx$

Example 10: Evaluating Definite Integrals using Substitution

Evaluate $\int_{-2}^{6} x^2 \sqrt[3]{x+2}dx$

Example 11: Evaluating Definite Integrals using Substitution

Evaluate $\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x dx$