Section 10: The Average Value of a Function on an Interval

To calculate the average value of a function f defined on an interval $[a,b]$, we partition the interval into n equal subintervals with width $\Delta x = \dfrac{b-a}{n}$. We evaluate the function at each of the points $a = x_0 < x_1 < x_2 < \cdots < x_n$ and take the average of these to approximate the average value of the function. That is,

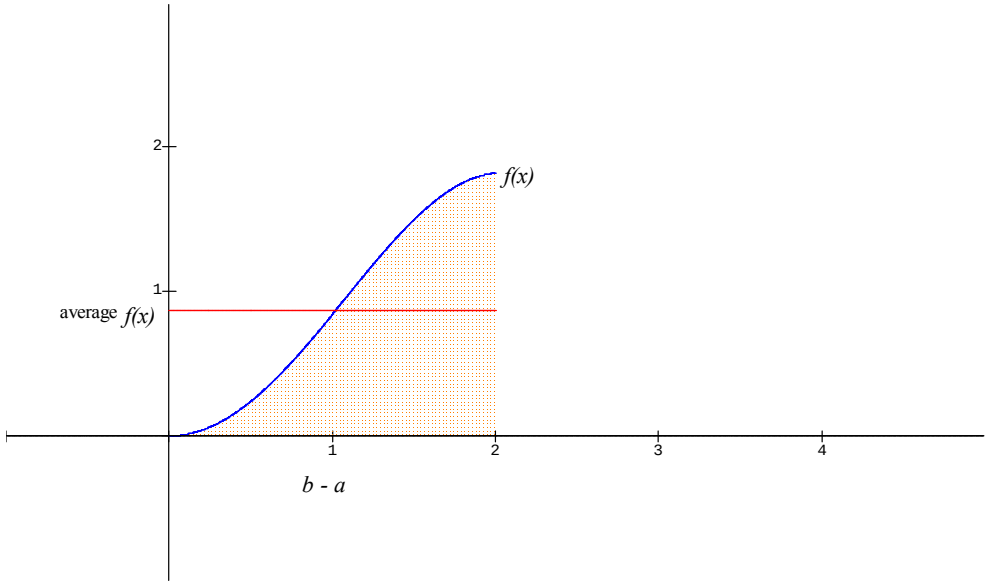
$$f_{ave} = \overline{f} \approx \frac{1}{n}\left[f(x_1) + f(x_2) + \cdots + f(x_2)\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right) = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x$$

If we take the limit of this last expression as $n \to \infty$, we have

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)dx\,.$$

There is an easy way to think of the average value of a function graphically. We know that on an interval $[a,b]$, the integral $\int_a^b f(x)dx$ can be thought of as the area under the function (if $f(x) \geq 0$). The area under the graph of the function from $[a,b]$ is the same as the area of the rectangle with length $b-a$ and height $\overline{f}$: $\int_a^b f(x)dx = \overline{f}\,(b-a)$.



Example 1: Finding the Average Value of a Function

Find the average value of the function $f(x) = x^2$ over the interval $[0, 2]$.

Example 2: Finding the Average Value of a Function

Find the average value of the function $f(x) = \cos x$ over the interval $[0, 2\pi]$.

Example 3: **Finding the Average Value of a Function**

Find the average value of the function $f(x) = 2x + 1$ over the interval $[0, 2\sqrt{2}]$.

Example 4: **Finding the Average Value of a Function**

Find the average value of the function $f(x) = \sqrt{1 - x^2}$ over the interval $[-1, 1]$.

Example 5: **Finding the Average Value of a Function**

Find the average value of the function $f(x) = x \sin x$ over the interval $[0, 2]$.

Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number $c \in [a, b]$ such that

$$f(c) = \overline{f} = \frac{1}{b-a} \int_a^b f(x)dx$$

that is,

$$\int_a^b f(x)dx = f(c)(b-a)$$