Work

For any constant force F applied over a distance d, we define the work ***W*** done as

$$W = Fd.$$

In the SI, mass is measured in kilograms (*kg*), distance is measured in meters (*m*), time in seconds (*s*), and the force in newtons (*N*), and $1N = 1kg \cdot 1m/s^2$. The SI unit of work is the joule (*J*). $1J = 1N \cdot m \ (\text{newton} \cdot \text{meter})$.

Example 1: Work done using a constant force

If we lift a mass of 3.25 *kg* straight up to a distance of 2.25 *m* against the acceleration of gravity of $9.81m/s^2$ requires a force

$$F = 3.25 \text{ kg} \cdot 9.81 \,{}^{\text{m}}\!/\!{}_{\text{s}^2} \approx 31.88N$$

The work done in moving the object is

$$Fd = 31.88N \cdot 2.25 \text{ m} = 71.73 \text{ J}$$

Next, consider a non-constant force $F = f(x)$ applied on an interval $[a,b]$. We first divide the interval $[a,b]$ into *n* subintervals of equal length $\Delta x = \dfrac{b-a}{n}$ and consider the work done on each subinterval. If Δx is small, then the force $f(x)$ applied on the subinterval $[x_{i-1}, x_i]$ can be approximated by the constant force $f(c_i)$ for some point $c_i \in [x_{i-1}, x_i]$. The work done on the subinterval is then approximated by $f(c_i)\Delta x$. The total work ***W*** is then approximately

$$W \approx \sum_{i=1}^{n} f(c_i)\Delta x$$

As we divide the interval into more subintervals, we have $n \to \infty$, and

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)dx$$

Hooke's Law

The force required to maintain a spring in a given position is proportional to the distance it is compressed or stretched. So, if x is the distance a spring is compressed (or stretched) from its natural length, the force $f(x)$ exerted by the spring is given by

$$f(x) = kx$$

for some constant k, which is called the **spring constant**.

Example 2: Work done using Hooke's Law

A force of 5 pounds is required to hold a spring that has been stretched $\frac{1}{3}$ foot from its natural length. Find the work done in stretching the spring 8 inches beyond its natural length.

Solution: we use the formula $f(x) = kx$ with $f(x) = 5$ and $x = \frac{1}{3}$, so that $5 = k\frac{1}{3}$. This gives $k = 15$.

So the spring constant is 15 and the force required to stretch or compress the string is $f(x) = 15x$.

The work done in stretching the string 8 inches, $\left(2/3\right)$ foot, is then

$$W = \int_a^b f(x)dx = \int_0^{2/3} 15x dx$$

$$= \left[\frac{15}{2}x^2\right]_0^{2/3} = \frac{10}{3} \text{ foot pounds}$$

Example 3: Work done using a variable force

A force $f(x)=\dfrac{15}{x+5}N$ where x denotes the distance in meters from the starting point, moves an object along the x-axis. Find the total work, in joules, expended in moving the object from $x=1$ to $x=4$.

Solution: $W=\int_a^b f(x)dx=\int_1^4 \dfrac{15}{x+5}dx=15\left[\ln|x+5|\right]_1^4=15\ln\left|\dfrac{3}{2}\right|\approx 6.08J$

Fluid Pressure

Definition: If a force of magnitude F is applied to a surface of area A, then the **pressure** P exerted by the force on the surface is $P=\dfrac{F}{A}$

Definition: If a fluid is homogeneous then its **mass density** is a constant $\delta=\dfrac{m}{V}$.

Sometimes we use **weight density** instead which is weight per unit volume: $\rho=\dfrac{W}{V}$ where W is the weight of a fluid of volume V.

Theorem: The fluid force and pressure on a flat surface that is submerged horizontally at depth h are given by

$$F=\rho hA$$
$$P=\rho h$$

Example 4: Fluid Pressure and Force

Find the fluid pressure and force on the top of a flat circular plate of radius 2 m that is submerged horizontally in water at a depth of 6 m. Hint: The weight density of fresh water is $9{,}810\ N/m^3$.

Solution:

Theorem: Suppose that a flat surface is submerged vertically in a fluid of weight density ρ and that the submerged portion of the surface extends from $x = a$ to $x = b$ along an $x-$axis whose positive direction is down. For $a \le x \le b$, suppose that $w(x)$ is the width of the surface and that $h(x)$ is the depth of the point x. Then, we define the fluid force F on the surface to be $F = \int_a^b \rho h(x) w(x) dx$

Example 5: Fluid Pressure

A dam is shaped like a trapezoid with height 60 ft. The width at the top is 100 ft and the width at the bottom is 40 ft.

a) Find the maximum hydrostatic force that the dam will need to withstand.
b) What happens to the force is the water level drops 10 ft?

Example 6: Work

A spherical tank of radius 10 ft is filled with water. Find the work done in pumping all the water out through the top of the tank. Hint: Water weighs $62.4\ \text{lb/ft}^3$.