Section 2 Integration by Parts

The Derivation of the Integration by Parts Formula

The technique of integration by parts comes from the formula for the derivative of a product:

$$\left[f(x)g(x)\right]' = f'(x)g(x) + f(x)g'(x)$$

If we integrate both sides of the equation and use the fact that

$$\int [f(x) + g(x)]\,dx = \int f(x)dx + \int g(x)dx\,,$$

we have

$$\int \left[f(x)g(x)\right]' dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

Subtracting $\int f'(x)g(x)dx$ from both sides gives

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

If we let $u = f(x)$ and $dv = g'(x)dx$, then $du = f'(x)dx$ and $v = g(x)$ and the **integration by parts formula** can be written in the easy to remember form:

$$\int u\,dv = uv - \int v\,du$$

By making the appropriate choice for *u* and *dv* we can write certain integrals in this form.

As a rule of thumb, you might want to use the acronym LIPTE to remember which function to choose as *u*. A good choice for *u* is the type of function that appears first in LIPTE.

Logarithmic
Inverse Trigonometric
Polynomial (algebraic)
Trigonometric
Exponential

Example 1: Integration by Parts

Evaluate $\int x \sin x dx$ using integration by parts.

Solution: Choose $u = x$ and $dv = \sin x dx$,
then $du = dx$ and $v = -\cos x$.

The integral then becomes

$$\int x \sin x dx = -x \cos x - \int -\cos x dx = -x \cos x + \sin x + c$$

Example 2: Integration by Parts

Evaluate $\int \ln x dx$ using integration by parts.

Example 3: Integration by Parts

Evaluate $\int x^2 \ln x dx$ using integration by parts.

Example 4: Integration by Parts

Evaluate $\int x^2 \sin x dx$ using integration by parts.

Example 5: Integration by Parts

Evaluate $\int xe^{2x}\,dx$ using integration by parts.

Example 6: Integration by Parts

Evaluate $\int x^2 e^{-3x}\,dx$ using integration by parts.

Example 7: Integration by Parts

Evaluate $\int e^{2x} \sin x dx$ using integration by parts.

Example 8: Integration by Parts

Evaluate $\int \tan^{-1} x dx$ using integration by parts.

Exercises:

a) $\int \cos x \cos 2x dx$

b) $\int x \sec^2 x dx$

c) $\int (\ln x)^2 dx$

d) $\int \cos x \ln(\sin x) dx$

Definite Integrals and Integration by Parts

To compute definite integrals with integration by parts, we start with the derivative of a product formula once again but take definite integrals on both sides:

$$\int_a^b \left[f(x)g(x)\right]' dx = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx$$

$$f(x)g(x)\Big|_a^b = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx$$

It follows that

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx$$

In *udv* form, we have $\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$.

Example 9: Integrating a Definite Integral by Parts

Evaluate $\int_1^2 x^2 \ln x dx$ using integration by parts.

Example 10: Integrating a Definite Integral by Parts

Evaluate $\int_1^{10} \ln x dx$ using integration by parts.

Example 11: Integrating a Definite Integral by Parts

Evaluate $\int_0^1 x \sin \pi x dx$ using integration by parts.

Integration by Parts – The Tabular Method

There is a shortcut method for calculating integrals that involve repeated use of integration by parts. The method is referred to as the tabular method since it involves creating a table.

Example 12: Integration by Parts Using the Tabular Method

Integrate $\int x \sin x dx$

Solution: Choose $u = x$ and $dv = \sin x$,

Make a Table -

u	dv	± 1
x	*sinx*	+1
1	*-cosx*	-1
0	*-sinx*	+1
		-1

Differentiate Integrate

Multiply across the diagonals as shown. The answer is then

$$\int x \sin x dx = -x \cos x + \sin x + C$$

Don't forget to add *C*!

Example 13: Integration by Parts Using the Tabular Method

Integrate $\int x^3 \ln x dx$ using the tabular method of integration by parts

Solution: Choose $u = \ln x$ and $dv = x^3$,

u	dv	± 1
$\ln x$	x^3	+1
$\frac{1}{x}$	$\frac{x^4}{4}$	-1

$$\int x^3 \ln x dx = \frac{1}{4} x^4 \ln x - \int \frac{1}{4} x^3 dx$$

Example 14: Integration by Parts Using the Tabular Method

Integrate $\int x^2 \sin x dx$ using the tabular method of integration by parts

Solution: Choose $u = x^2$ and $dv = \sin x$,

u	dv	± 1
x^2	*sinx*	+1
$2x$	*-cosx*	-1
2	*-sinx*	+1
0	$\cos x$	-1
		+1

$$\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

Example 15: Integration by Parts Using the Tabular Method

Integrate $\int x^2 \sin x dx$ using the tabular method of integration by parts

Solution:

u	dv	± 1
$\sin x$	e^x	+1
$\cos x$	e^x	-1
$-\sin x$	e^x	+1
	e^x	-1
		+1

$$\int e^x \sin x dx = e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

$$2\int e^x \sin x dx = e^x \sin x - e^x \cos x$$

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$