Section 4 Approximating Values of Definite Integrals

The definite integral $\int_a^b f(x)dx$ is, by definition, a limit of sums of the form $\sum_{i=1}^{n} f(c_i)\Delta x$.

Consequently, *any* such sum provides an estimate of $\int_a^b f(x)dx$. In this section, we will develop three methods for approximating definite integrals: the trapezoidal rule, Simpson's rule, and the midpoint rule. These techniques are especially useful since many elementary functions do not have elementary antiderivatives.

The Midpoint Rule

Let *n* be a positive integer. Divide the interval $[a,b]$ into *n* sections of equal length $\Delta x = \frac{b-a}{n}$. The midpoint of each interval $[x_{i-1}, x_i]$ is given by $\overline{x}_i = \frac{(x_{i-1}+x_i)}{2}$. The midpoint rule approximation for the integral $\int_a^b f(x)dx$ is given by

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\Delta x =$$

$$M_n = \Delta x\left[f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + \cdots + f(\overline{x}_{n-1}) + f(\overline{x}_n)\right]$$

Example 1: Using the Midpoint Rule

Find the midpoint approximation of $\int_0^2 x^2 dx$ using $n = 4$.

Solution: With $n = 4$, we get $\Delta x = \frac{2-0}{4} = \frac{1}{2}$. This gives us the partition of the interval from 1 to 2 into 4 subintervals: $\left[0,\frac{1}{2}\right]$, $\left[\frac{1}{2},1\right]$, $\left[1,\frac{3}{2}\right]$, and $\left[\frac{3}{2},2\right]$. The midpoints of these intervals are $\overline{x}_1 = \frac{1}{4}$, $\overline{x}_2 = \frac{3}{4}$, $\overline{x}_3 = \frac{5}{4}$, and $\overline{x}_4 = \frac{7}{4}$. Using these values, we get

$$\begin{aligned} M_4 &= \Delta x\left[f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + f(\overline{x}_4)\right] \\ &= \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right)\right] \\ &= \frac{1}{2}\left[\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{5}{4}\right)^2 + \left(\frac{7}{4}\right)^2\right] \\ &= \frac{1}{2}\left[\frac{1}{16} + \frac{9}{16} + \frac{25}{16} + \frac{49}{16}\right] = \frac{84}{32} = 2.625 \end{aligned}$$

Error Bound: Suppose $|f''(x)| \le K$ for $a \le x \le b$, the error bound in the midpoint estimate E_M is given by $|E_M| \le \dfrac{K(b-a)^3}{24n^2}$.

Example 2: Error Bound Using the Midpoint Rule

Find the error bound for the approximation in example 1.

The Trapezoidal Rule

The trapezoidal rule estimates the area under a curve by approximating $f(x)$ with line segments on the interval $[a,b]$. This creates trapezoids whose areas will sun to be about equal to $\int_a^b f(x)dx$. Let *n* be a positive integer. Divide the interval $[a,b]$ into *n* subintervals of equal length $\Delta x = h = \dfrac{b-a}{n}$. Each ith subinterval has area with integration nodes

$$x_0 = a \quad , x_1 = a + h, \; x_2 = a + 2h, \; \ldots, \; x_n = b.$$

The sum $\dfrac{f(x_0)+f(x_1)}{2}h + \dfrac{f(x_1)+f(x_2)}{2}h + \cdots + \dfrac{f(x_{n-1})+f(x_n)}{2}h$ is called the trapezoidal estimate of $\int_a^b f(x)dx$. It is usually written

$$T_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Notice that the first and last values have coefficient 1 and all the others have coefficient 2.

If *f* is concave downward on $[a,b]$, then the trapezoidal method underestimates the integral $\int_a^b f(x)dx$.

Error Bound: Suppose $|f''(x)| \le K$ for $a \le x \le b$, the error bound in the trapezoidal estimate E_T is given by

$$|E_T| \le \frac{K(b-a)^3}{12n^2}$$

Example 3: Using the Trapezoidal Rule

Find the trapezoidal approximation of $\int_0^2 x^3 dx$ using $n = 4$. Find the bound in the error.

Simpson's Rule

Let n be an even positive integer. Divide the interval $[a,b]$ into n sections of equal length $h = \frac{b-a}{n}$. This gives the integration nodes

$$x_0 = a \quad , x_1 = a + h,\ x_2 = a + 2h,\ \ldots,\ x_n = b.$$

The sum $S_n = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$ is called the

Simpson's estimate of $\int_a^b f(x)dx$.

Notice that the first and last values have coefficient 1 and all the others have coefficients alternating 4, 2, 4, 2, …, 2, 4.

Error Bound: Suppose $\left|f^{(4)}(x)\right| \le K$ for $a \le x \le b$, the error bound in the Simpson's estimate E_S is given by

$$\left|E_S\right| \le \frac{K(b-a)^5}{180n^4}$$

Example 4: Using Simpson's Rule

Find the Simpson's rule approximation of $\int_0^2 x^3 dx$ using $n = 4$. Find the bound in the error.