**Section 5 Improper Integrals**

**Type 1: Infinite Intervals**

**Definition: An Improper Integral of Type 1**

a) If $\int_a^b f(x)dx$ exists for every number $b \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx \text{ provided this limit exists.}$$

b) If $\int_a^b f(x)dx$ exists for every number $a \leq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx \text{ provided this limit exists.}$$

In either case, if the limit exists then the integral **converges,** otherwise it is said to be **divergent.**

**Example 1: Evaluating an Improper Integral**

Evaluate if possible:

a) $\int_1^\infty \frac{1}{x} dx$

b) $\int_1^\infty \frac{1}{x^2} dx$

**Example 2:** **Evaluating an Improper Integral**

Evaluate: $\int_{-\infty}^{0} xe^{-x}dx$

**Example 3:** **Evaluating an Improper Integral**

Evaluate: $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

**Example 4:** **Evaluating an Improper Integral**

For what values of p does the integral $\int_1^{\infty} \frac{1}{x^p}\, dx$ converge?

**Type 2: Discontinuous Integrands**

**Definition: An Improper Integral of Type 2**

c) If $f$ is continuous on $[a,b)$ and discontinuous at $b$, then

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx \text{ provided this limit exists.}$$

d) If $f$ is continuous on $(a,b]$ and discontinuous at $a$, then

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx \text{ provided this limit exists.}$$

In either case, if the limit exists then the integral **converges,** otherwise it is said to be **divergent.**

**Example 5: Evaluating an Improper Integral**

Evaluate $\int_2^5 \frac{1}{\sqrt{x-2}}dx$

**Example 6: Evaluating an Improper Integral**

Evaluate $\int_0^{\pi/2} \sec x dx$

**Example 7: Evaluating an Improper Integral**

Evaluate $\int_0^{3} \frac{1}{x-1} dx$

**The Comparison Test for Improper Integrals**

Suppose that $f$ and $g$ are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
2. If $\int_a^\infty f(x)dx$ is divergent, then $\int_a^\infty g(x)dx$ is divergent.

**Example 8: Using the Comparison Test for Integrals**

Determine whether the integral is convergent or divergent: $\int_1^\infty \frac{\cos^2 x}{1+x^2}dx$