Section 7 Review of Parametric Curves

Definitions: Let f and g be continuous functions of t on an interval I. The set of all points (x,y) where $x = f(t)$ and $y = g(t)$ is called a **plane curve**. The variable $\boldsymbol{t}$ is called a **parameter** and the equations defining x and y are called **parametric equations**. A pair of parametric equations is called a **parameterization** of the curve.

Example 1: Graphing Parametric Curves

Sketch and identify the curved defined by the parametric equations

$$x = t^2 - 2t \text{ and } y = t + 1$$

Definition: If the parameter t is defined on an interval $a \le t \le b$, then the curve

$$x = f(t), \quad y = g(t), \quad a \le t \le b$$

has an **initial point** $(f(a), g(a))$ and a **terminal point** $(f(b), g(b))$.

Example 2: Identifying Parametric Curves

What curves are defined by each set of parametric equations?

a) $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

b) $x = \sin 2t$, $y = \cos 2t$, $0 \le t \le 2\pi$

Example 3: Graphing Parametric Curves

Sketch the curve with parametric equations $x = \sin t$, $y = \sin^2 t$.

Example 4: Graphing Parametric Curves

Sketch the curve by using the parametric equations to plot points. $x = e^{-t} + t$, $y = e^t - t$, $-2 \le t \le 2$

Example 5: Graphing Parametric Curves

Sketch the curve by using the parametric equations to plot points. Indicate the direction the curve is traced out by using an arrow. $x = t^2$, $y = 6 - 3t$.

Eliminate the parameter to find a Cartesian equation of the curve.

Example 6: Finding Cartesian equations of Parametric Curves

Eliminate the parameter to find a Cartesian equation of the curve.

$$x = e^t,\ y = e^{-t}$$

Example 7: Finding Cartesian Equations of Parametric Curves

Eliminate the parameter to find a Cartesian equation of the curve.

$$x = 4\cos\theta,\ y = 5\sin\theta,\ \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$$

Tangents to Parametric Curves

Let f and g be differentiable functions of t on an interval I with $x = f(t)$ and $y = g(t)$. The tangent line to the curve defined by these parametric equations is found using the chain rule. Assuming y is also a differentiable function of x, we have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Solving for $\frac{dy}{dx}$, we see that $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ as long as $\frac{dx}{dt} \neq 0$.

Example 8: Finding the tangent line to a parametric curve

Find an equation of the tangent line to the parametric curve $x = 2\sin 2t$, $y = 2\sin t$ at the point $(\sqrt{3}, 1)$. Where does the curve have horizontal tangents?

Example 9: Finding the tangent line to a parametric curve

Find an equation of the tangent line to the curve with parametric equations $x = t\sin t$, $y = t\cos t$ at the point $(0, -\pi)$.

Area Enclosed by Parametric Curves

The area under $y = F(x)$ from a to b is given by $A = \int_a^b F(x)dx$. If the curve is defined parametrically by the parametric equations $x = f(t)$ and $y = g(t)$ with $\alpha \le t \le \beta$ and the curve is traced out once as x goes from α to β, then the area can be found using

$$A = \int_a^b y dx = \int_\alpha^\beta g(t) f'(t) dt .$$

Example 10: Finding the Area enclosed by a Parametric Curve

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a, b > 0$.

The ellipse has parametric equations $x = a\cos\theta$, $y = b\sin\theta$. We use symmetry to get $A = 4\int_0^a y dx = 4\int_{\pi/2}^0 b\sin\theta(-a\sin\theta)d\theta = ab\pi$

Example 11: Finding the Area enclosed by a Parametric Curve

Find the area under one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

Solution

$A = \int_0^{2\pi r} y dx = \int_0^{2\pi} r(1 - \cos\theta) r(1 - \cos\theta) d\theta = 3\pi r^2$