Section 8 Volumes Generated by Solids of Revolution

If the cross-sectional area, $A(x)$, varies continuously with *x*, then we can find the volume by integrating from *a* to *b*.

$$V = \int_a^b A(x)dx .$$

Example 1: Finding Volumes Using Cross-Sections

Find the volume of a pyramid whose altitude is *h* and whose base is a square with sides of length *r*.

Volume Using Discs

If f is nonnegative and continuous on the interval $[a, b]$, then the volume generated by rotating the region bounded by the x-axis and the graph of f about the x-axis is given by

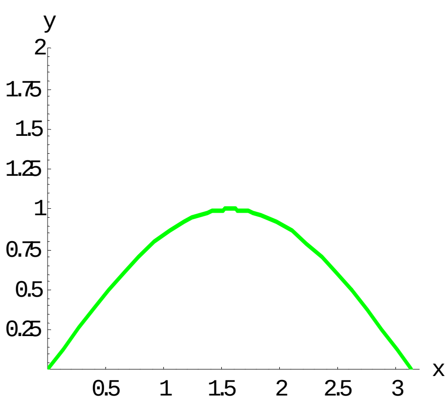
$$V = \int_a^b \pi [f(x)]^2 \, dx \, .$$

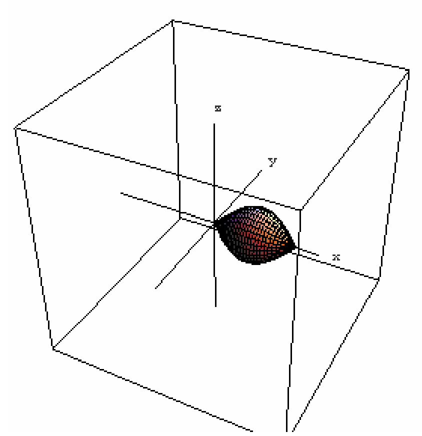
Similarly, if g is nonnegative and continuous on $[c, d]$, then the volume generated by rotating the region bounded by the y-axis and graph of g about the y-axis is given by

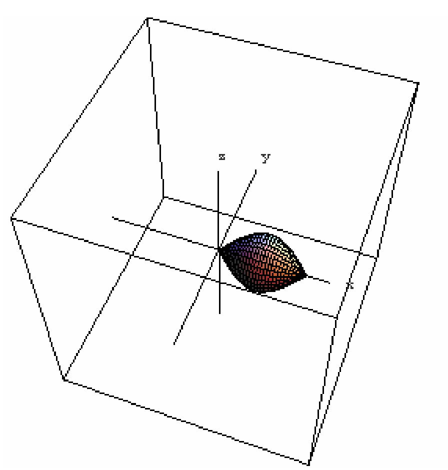
$$V = \int_c^d \pi [g(y)]^2 \, dy \, .$$

Example 2: Finding the volume of a solid of revolution using discs

Find the volume generated by revolving the region bounded by the function $f(x) = \sqrt{\sin x}$ and the lines $x = 0$ and $x = \pi$ about the x-axis.







The volume is given by

$$V = \int_a^b \pi [f(x)]^2 \, dx = \int_0^\pi \pi \left[\sqrt{\sin(x)}\right]^2 dx$$

$$= \pi \int_0^\pi \sin(x) dx = -\pi \cos(x)\Big|_0^\pi$$

$$= -\pi \cos(\pi) + \pi \cos(0) = 2\pi$$

Example 3: Finding the volume of a solid of revolution using discs

Find the volume generated by revolving the region bounded by the function $f(x) = x^2 + 1$ and the lines $x = -1$ and $x = 1$ about the *x*-axis.

Example 4: Finding the volume of a solid of revolution using discs

Find the volume generated by revolving the region bounded by the function $y = x^3$ and the lines $y = 1$ and $y = 8$ about the *y*-axis.

Volume Using Washers

Suppose that *f* and g are nonnegative and continuous functions with $g(x) \leq f(x)$ for all x in $[a, b]$, then the volume generated by rotating the region bounded by *f* and *g* about the *x*-axis is given by

$$V = \int_a^b \pi\left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right) dx\,.$$

If a region is revolved around the *y*-axis, then we get an integral of the form:

$$V = \int_c^d \pi\left(\left[F(y)\right]^2 - \left[G(y)\right]^2\right) dy$$

Example 5: Finding the volume of a solid of revolution using washers

Find the volume generated by revolving the region in the first quadrant bounded by the function $y = \frac{1}{8}x^3$ and the lines $y = 2x$ and $y = 8$ about the *y*-axis.

Example 6: Finding the volume of a solid of revolution using washers

Find the volume generated by revolving the region bounded by the equations $x^2 = y - 2$, $2y - x - 2 = 0$, $x = 0$, and $x = 1$ about the *x*-axis.

Volume Using Shells

Suppose that *f* and g are nonnegative and continuous functions with $g(x) \leq f(x)$ for all *x* in $[a, b]$, then the volume generated by rotating the region bounded by *f* and *g* about the *y*-axis is given by

$$V = \int_a^b 2\pi x\left([f(x)] - [g(x)]\right) dx .$$

If a region is revolved around the *x*-axis, then we get an integral of the form:

$$V = \int_c^d 2\pi y\left([F(y)] - [G(y)]\right) dy .$$

If the region is revolved about a line other than an axis, then an integral of the form:
$V = \int_a^b 2\pi r\left([f(x)] - [g(x)]\right) dx$ or $V = \int_c^d 2\pi r\left([F(y)] - [G(y)]\right) dy$ will result.

Example 7: Finding the volume of a solid of revolution using shells

Find the volume generated by revolving the region between the curves $y = x^2$ and $y = x^3$ about the y-axis.

Example 8: Finding the volume of a solid of revolution using shells

Find the volume generated by revolving the region bounded by the graphs of $y = 2x - x^2$ and the x-axis about the y-axis.

Example 9: Finding the volume of a solid of revolution using shells

Find the volume generated by revolving the region in the first quadrant bounded by the graphs of $x = 2y^3 - y^4$ and the y-axis about the x-axis.