Section 9 Arc Length

Definition: A **smooth curve** has parametric representation $x = f(t)$, $y = g(t)$, on an interval *I* such that $f'(t)$ and $g'(t)$ are continuous and not simultaneously zero except possibly at the endpoints.

Theorem: If a smooth curve *C* is given parametrically by $x = f(t)$, $y = g(t)$, and, where $a \le t \le b$, and if *C* does not intersect itself, except possibly at the endpoints of $[a,b]$, then the length of *C* is given by

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

or

$$= \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt .$$

If the curve is expressed in rectangular coordinates then we can use the parameterization $x = x$, $y = f(x)$, then the formula becomes

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Example 1: Finding the Arc Length of a Linear Function

Find the arc length of the line segment on $f(x) = mx + b$ from (x_1, y_1) to (x_2, y_2).

Example 2: Finding the Arc Length of a Function

Find the arc length of the graph of $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $\left[\frac{1}{2}, 2\right]$.

Example 3: Finding the Arc Length of a Function

Find the arc length of the graph of $(y-1)^3 = x^2$ on the interval $[0, 8]$. Hint: Solve for y first.

Example 4: Finding the Arc Length of a Parametric Function

Find the arc length of the graph of the circle parameterized by the equations $x(t) = a\cos t$ and $y(t) = a\sin t$ with $t \in [0, 2\pi]$.

Example 5: Finding the Arc Length of a Parametric Function

Find the arc length of the graph of the curve parameterized by the equations $x(\theta) = a(\cos\theta + \theta\sin\theta)$ and $y(\theta) = a(\sin\theta - \theta\cos\theta)$ with $\theta \in [0, \pi]$.