

Differential Equations

Definition: A relationship between a function and its derivatives is called a **differential equation**. The highest-order derivative that appears is called the **order** of the differential equation. The general form of an **n th-order differential equation** with independent variable x and unknown function $y(x)$ is $f(x, y, y', y'', \dots, y^{(n)}) = 0$ where f is a real-valued function of $n + 2$ variables.

Example 1:

- a) The equation $\frac{dy}{dx} = ky$ is a first-order differential equation.
- b) The equation $\frac{d^2y}{dx^2} + y = 0$ is a second-order differential equation

These differential equations are called **ordinary differential equations** because the unknown function depends on a single independent variable. If the unknown function depends upon more than one independent variable, then the equation is called a **partial differential equation**.

Example 2: The equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{k} \frac{\partial y}{\partial t}$ is a partial differential equation because the unknown function y depends on both x and t .

We will have three important goals in our study of differential equations:

1. To discover the differential equation that describes a physical situation (modeling).
2. Find the solution – exact or approximate - of the equation (solving).
3. Interpret the solution (interpreting).

Example 3: The equation $\frac{dP}{dt} = kP$ is a first-order differential equation that is used to model the time rate of change of population $P(t)$ with constant birth and death rates. This equation is used because under those conditions the rate of change is proportional to the population.

Definition: We say that a function y **satisfies** or is a **solution** of the differential equation on an interval I if the function has the indicated derivatives and makes the equation true for all values of the independent variable in the interval I .

One nice aspect of the study of differential equations is that one can always verify that a given function is a solution by taking derivatives and substituting into the original equation and verifying that an identity results.

Example 4: The function $P(t) = Ce^{kt}$ is a solution of the differential equation $\frac{dP}{dt} = kP$.

To see this we find the derivative of the function $P(t) = Ce^{kt}$ and substitute into the equation.

$$P'(t) = \frac{dP}{dt} = Cke^{kt} = k(Ce^{kt}) = kP.$$

Notice that each choice of the arbitrary constant C gives us a different solution. Thus the solution defines an infinite family of solutions.

Example 5: Verifying that a Given Function is a Solution of an ODE

Show that the equation $\frac{dy}{dt} = -0.04y + 32$ has for one solution the function

$$y(t) = 800 + 70e^{-0.04t}.$$

Example 6: Verifying that a Given Function is a Solution of an ODE

Verify that the functions $y_1 = e^t \cos t$ and $y_2 = e^t \sin t$ are both solutions of the differential equation $y'' - 2y' + 2y = 0$.

Note: It follows that the sum $y_1 + y_2$ is also a solution of the differential equation. Why?

As we have seen, an equation may have more than one solution. If we wish to have a specific solution, then we must have an initial condition – a condition on the value of the solution of the equation at some point.

Definition: A first-order equation together with an initial condition is called an **initial value problem**. A solution of the differential equation that also satisfies the initial condition is a solution of the initial value problem. A general first-order initial value problem is denoted by $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$.

Exercise 1: Verify that $y(x)$ satisfies the given differential equation. Then find a value of C so that $y(x)$ satisfies the initial value problem.

a) $y' = 2y; y(t) = Ce^{2t}; y(0) = 3$

b) $y' = t - y; y(t) = Ce^{-t} + t - 1; y(0) = 10$

c) $e^y y' = 1; y(t) = \ln(t + C); y(0) = 0$

In this class we will learn to solve many types of equations. There are, however, three basic types of first-order linear equations that we will solve. The simplest type of equation we can solve is of the form $\frac{dy}{dt} = f(t)$. In this type of equation, the dependent variable does not appear on the right-hand side of the equation. This equation can be solved directly by integrating both sides of the equation (if possible). An equation of the form $\frac{dy}{dt} = g(y)$ is called an autonomous differential equation since the independent variable does not appear in the right-hand side. The third type of equation is equations of the form $\frac{dy}{dt} = f(t, y)$.

Definitions: Consider the first-order differential equation of the form $\frac{dy}{dt} = f(t)$. We can solve this equation by integrating both sides.

$$\begin{aligned} \frac{dy}{dt} &= f(t) \\ y &= y(t) = \int f(t)dt + C \end{aligned} \quad (1.1)$$

Equation (1.1) defines the **general solution** of the differential equation $\frac{dy}{dt} = f(t)$. We can obtain a **particular solution** of the initial value problem $\frac{dy}{dt} = f(t)$, $y(t_0) = y_0$ by substituting $t = t_0$ and $y = y_0$ into the initial value problem and solving for C .

Example 7: Solving an Initial Value Problem

Solve the initial value problem $\frac{dy}{dt} = e^{-2t}$, $t > 0$, $y(0) = 5$.

Example 8: Solving an Initial Value Problem

Solve the initial value problem $\frac{dy}{dt} = t\sqrt{t^2 + 9}$, $y(-4) = 0$.

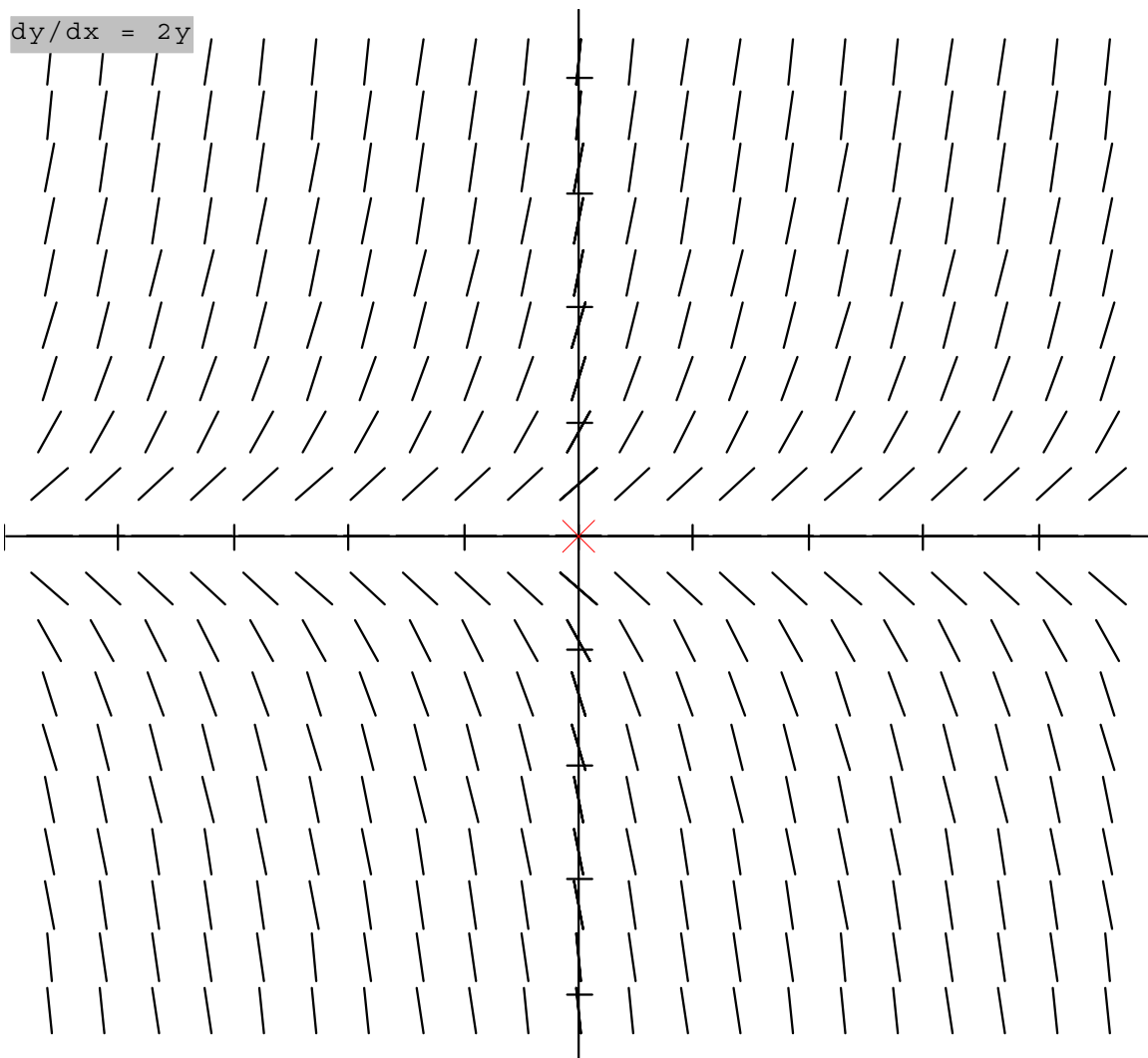
Exercise 2: Solving Initial Value Problems

a) Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1}{x(x+1)}$, $x > 0$.

b) Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1}{1+x^2}$.

c) Can you solve $\frac{dy}{dx} = e^{-x^2/2}$, $y(0) = 0$?

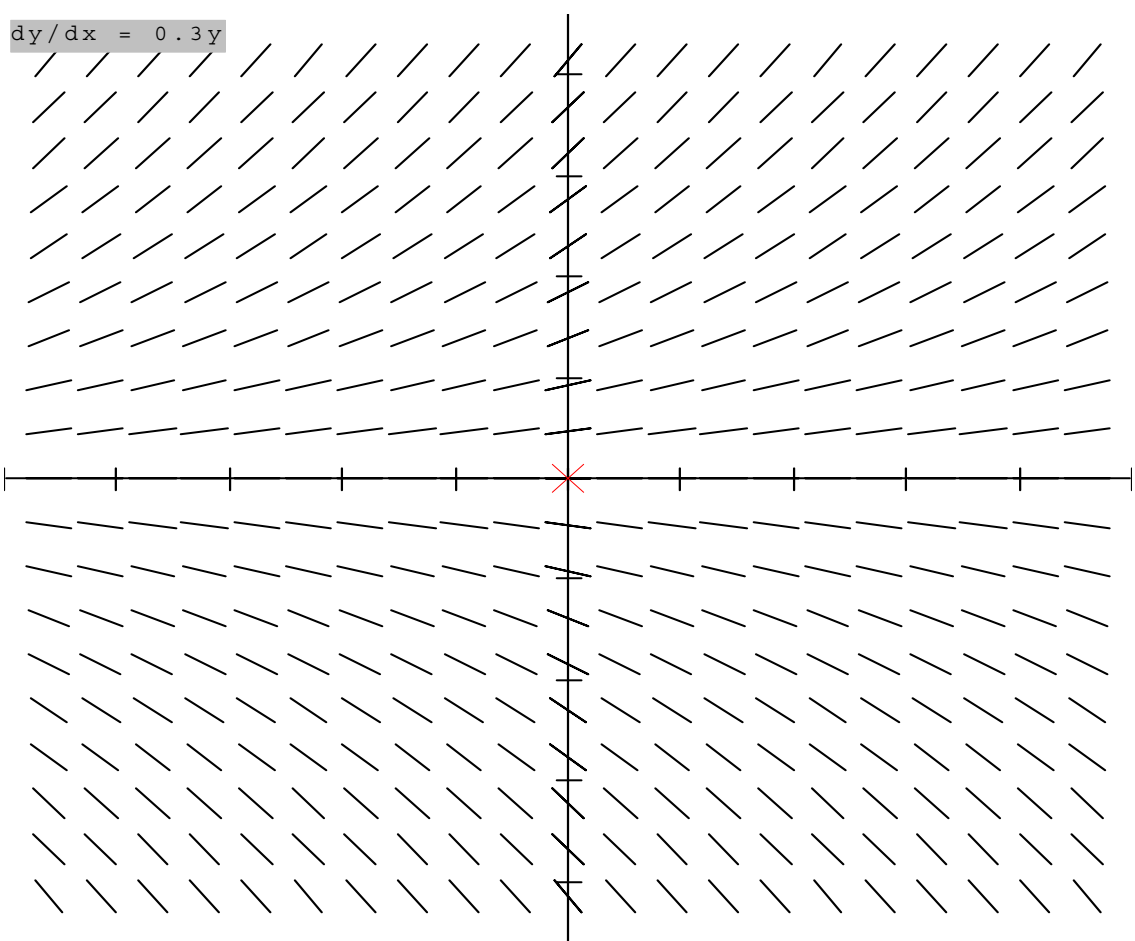
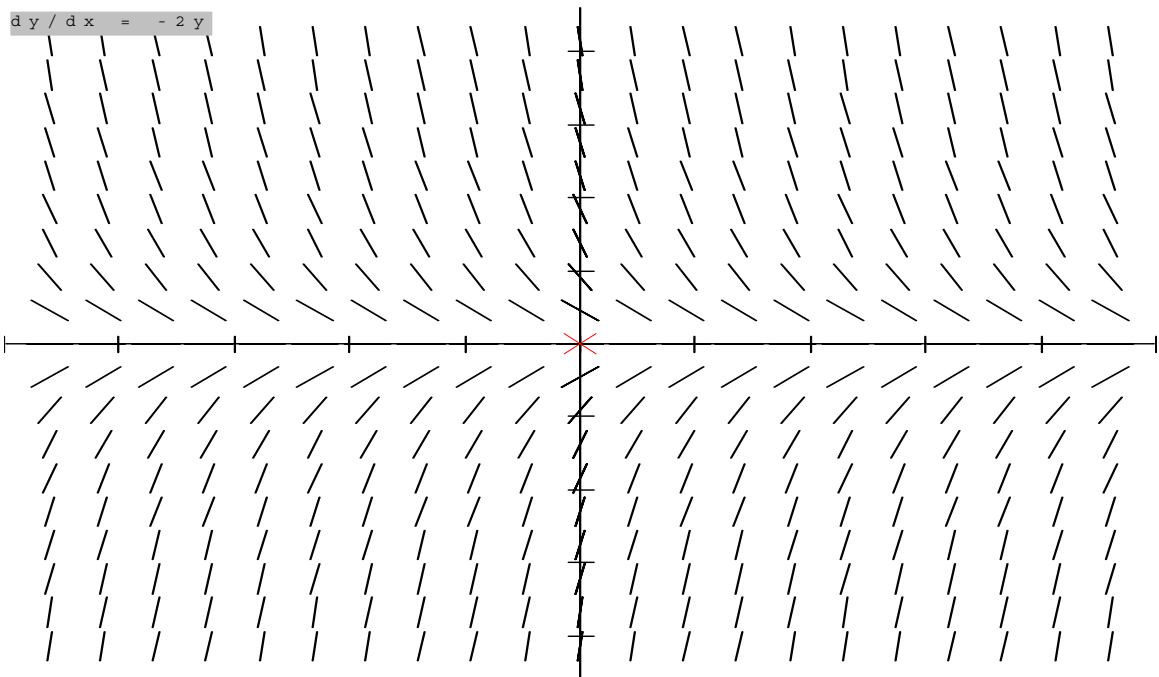
d) Can you solve $\frac{dy}{dx} + f(x)y = 0$?

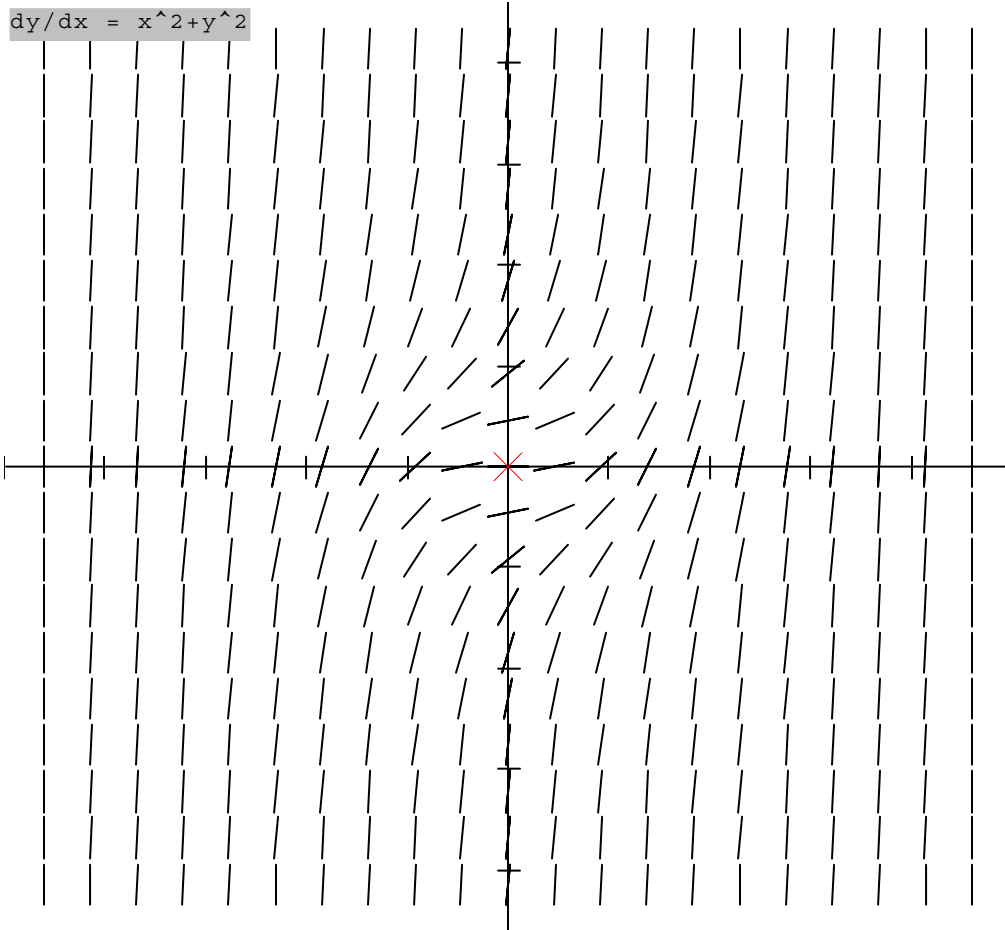


Definitions: Consider the differential equation $\frac{dy}{dx} = f(x, y)$. Through each of a representative sample of points (x, y) draw a short line segment having slope $m = f(x, y)$. The set of all these segments is called a **direction field** for the equation $\frac{dy}{dx} = f(x, y)$.

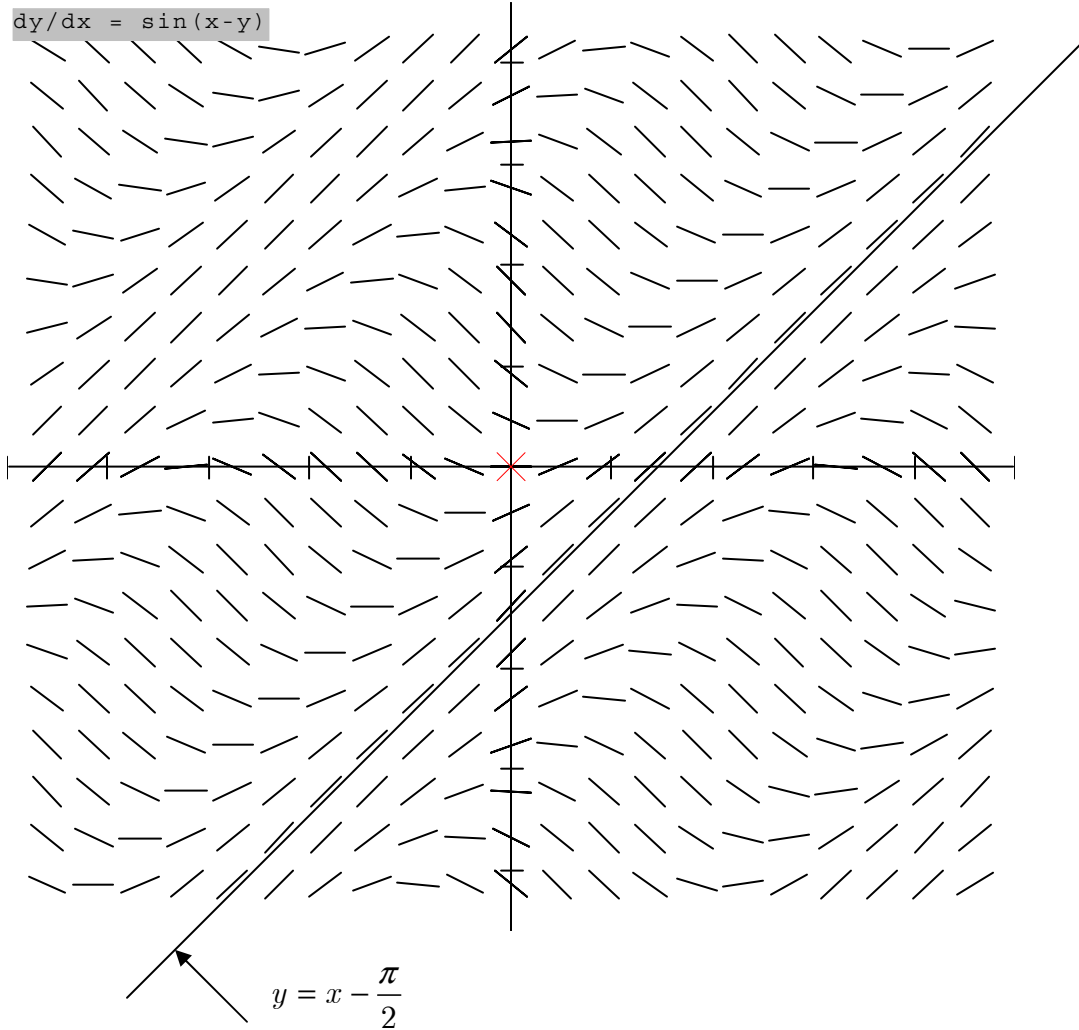
An **isocline** of the differential equation $\frac{dy}{dx} = f(x, y)$ is a curve of the form $f(x, y) = C$ on which the slope $y'(x)$ is a constant.

The isoclines of this curve are horizontal lines of the form $2y = C$ or $y = c$.



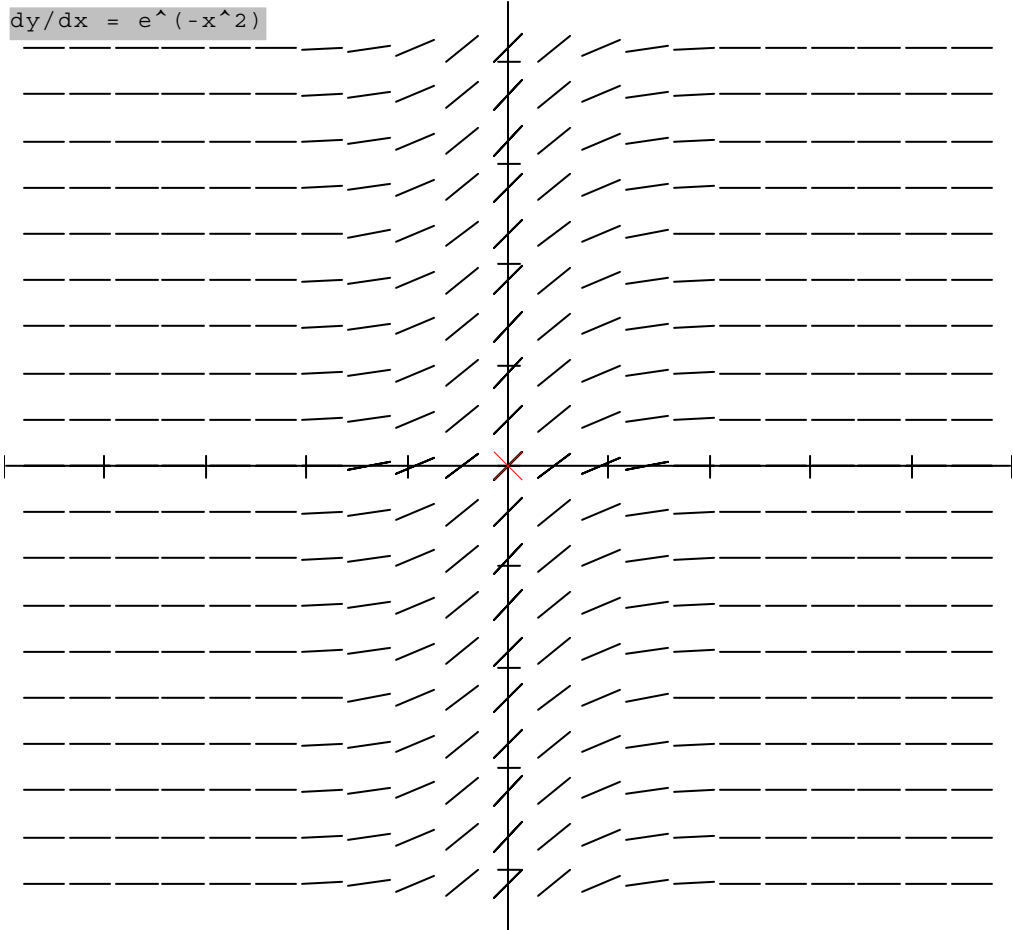


The isoclines of the differential equation $\frac{dy}{dx} = x^2 + y^2$ have equations $x^2 + y^2 = c > 0$ which are circles centered at $(0,0)$ with radius $r = \sqrt{c}$.



The isoclines of the differential equation $\frac{dy}{dx} = \sin(x-y)$ have equations $\sin(x-y) = c$ from which it follows that $x-y = \sin^{-1}(c)$ and $y = x - \sin^{-1}(c)$. Thus the isoclines are lines of slope 1.

$$dy/dx = e^{-x^2}$$



Separable Differential Equations

Definition: A differential equation that can be written in the form $g(y)\frac{dy}{dt} = f(t)$ is called a **separable differential equation**.

Using differentials we can write this equation in the form $g(y)dy = f(t)dt$ and then integrate both sides to get $\int g(y)dy = \int f(t)dt$. If $G(y)$ is any antiderivative of $g(y)$ and $F(t)$ is any antiderivative of $f(t)$, then the equation $G(y) = F(t) + C$ defines a family of solutions of the original equation implicitly. Under certain conditions we will be able to solve for the function $y(t)$ explicitly.

Example 1: Solving a separable differential equation

Find the general solution of the equation $\frac{dy}{dt} = \frac{t^2}{y^2}$

Example 2: Solving a separable differential equation

Find the general solution of the equation $e^y \frac{dy}{dt} - t - t^3 = 0$

Example 3: Solving an initial-value problem

Solve the initial-value problem $e^y \frac{dy}{dt} - t - t^3 = 0$, $y(1) = 1$

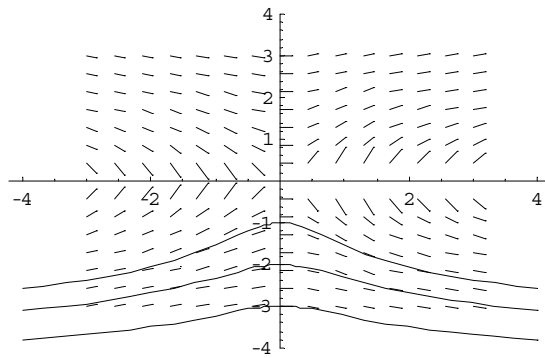
Example 4: Solving an initial-value problem

Solve the initial-value problem $(4y - \cos y) \frac{dy}{dx} - 3x^2 = 0$, $y(0) = 0$

Example 5: Solving an initial-value problem

$$y' = \frac{2x}{y + x^2y}, \quad y(0) = -2$$

- a) Solve the initial value problem in explicit form
- b) Plot the graph of the solution
- c) Determine the interval in which the solution is defined.



Mathematical Models

Unrestricted Growth Model

Rate of change is proportional to current population: $\frac{dP}{dt} = kP$

Radioactive Decay

Rate of change is proportional to amount present: $\frac{dQ}{dt} = -rQ$

Limited Growth Model

Rate of change is proportional to the difference between some maximum population M and the current population: $\frac{dP}{dt} = k(M - P)$

Logistic Growth Model

Rate of change is jointly proportional to the current population and the difference between some maximum population M and the current population: $\frac{dP}{dt} = kP(M - P)$

This models the spread of infectious diseases, the growth of a business, and the spread of a rumor.

Newton's Law of Cooling: The rate at which an object cools is proportional to the difference in temperature between the object and the surrounding environment: $\frac{dT}{dt} = k(T - T_a)$. We use T to represent the temperature of the object at any time t and T_a for the ambient temperature.

Radioactive Decay

If a radioactive substance A is not only decaying, but is also being formed by the decay of some other radioactive substance B, then the number N of nuclei of substance A present at time t is given by the differential equation: $\frac{dN}{dt} + \lambda N = \mu B_0 e^{-\mu t}$. Where λ and μ are the decay constants of A and B, respectively, and B_0 is the number of nuclei of B present at time $t = 0$. Note: We are making the assumption that there is only one decay route for the B nuclei so that the |rate of decrease of B| = |rate of increase of A|.

Electrical Circuit

An electrical circuit contains an inductor of inductance L , a resistor of resistance R , and an ideal voltage generator that produces an alternating voltage $E \cos(\omega t)$. The differential equation determining the electrical current in the circuit at time t is given by $L \frac{dI}{dt} + RI = E \cos(\omega t)$

