Name ____________________________ Test 1 Calculus 2 Mike Huff

1. $\int (x^{2/3} + \frac{x-4}{\sqrt{x^2-8x+1}})dx = \int x^{2/3}dx + \int \frac{x-4}{\sqrt{x^2-8x+1}}dx$

$= \frac{3}{5}x^{5/3} + \frac{1}{2}\int \frac{2x-8}{\sqrt{x^2-8x+1}}dx$. Let $u = x^2 - 8x + 1$, then $du = (2x-8)\,dx$.

$= \frac{3}{5}x^{5/3} + \int \frac{1}{2}u^{-1/2}du = \frac{3}{5}x^{5/3} + u^{1/2} + C$

$= \frac{3}{5}x^{5/3} + \left(x^2 - 8x + 1\right)^{1/2} + C = \frac{3}{5}x^{5/3} + \sqrt{x^2 - 8x + 1} + C$

2. $\int \tan^3 3x \sec^4 3x dx$

Method 1: Using *u*-substitution: Let $u = \sec 3x$, then $\frac{1}{3}du = \sec 3x \tan 3x dx$.

Then, $\int \tan^3 3x \sec^4 3x dx$

$= \int \tan^2 3x \sec^3 3x \sec 3x \tan 3x dx = \int \left(\sec^2 3x - 1\right)\sec^3 3x \sec 3x \tan 3x dx$

.

$= \frac{1}{3}\int \left(u^2 - 1\right)u^3 du = \frac{1}{3}\int \left(u^5 - u^3\right)du = \frac{1}{3}\left(\frac{u^6}{6} - \frac{u^4}{4}\right) + C$

$= \frac{1}{3}\left(\frac{\sec^6 3x}{6} - \frac{\sec^4 3x}{4}\right) + C$

Method 2: Using *u*-substitution: Let $u = \tan 3x$, then $\frac{1}{3}du = \sec^2 3x dx$.

Then, $\int \tan^3 3x \sec^4 3x dx$

$= \int \sec^2 3x \tan^3 3x \sec^2 3x dx = \int \left(\tan^2 3x + 1\right)\tan^3 3x \sec^2 3x dx$.

$= \frac{1}{3}\int \left(u^2 + 1\right)u^3 du = \frac{1}{3}\int \left(u^5 + u^3\right)du = \frac{1}{3}\left(\frac{u^6}{6} + \frac{u^4}{4}\right) + C$

$= \frac{1}{3}\left(\frac{\tan^6 3x}{6} + \frac{\tan^4 3x}{4}\right) + C$

3. $\int \sin 2x \cos 2x dx$

Method 1: Using *u*-substitution: Let $u = \cos 2x$, then $-\frac{1}{2}du = \sin 2x dx$.

Then,

$$\int \sin 2x \cos 2x dx = -\frac{1}{2}\int u du = -\frac{u^2}{4} + C = -\frac{\cos^2 2x}{4} + C$$

Method 2: Using *u*-substitution: Let $u = \sin 2x$, then $\frac{1}{2}du = \cos 2x dx$.

Then,

$$\int \sin 3x \cos 3x dx = \frac{1}{2}\int u du = \frac{u^2}{4} + C = \frac{\sin^2 2x}{4} + C$$

These two answers are actually the same!

4. $\int \tan^{-1}\frac{x}{2} dx$

Use integration by parts with $u = \tan^{-1}\frac{x}{2}$ and $dv = dx$ so that

$du = \frac{1}{2}\frac{1}{1+\left(\frac{x}{2}\right)^2}dx = \frac{2}{4+x^2}dx$ and $v = x$. Then,

$$\int \tan^{-1}\frac{x}{2} dx = x\tan^{-1}\frac{x}{2} - \int \frac{2x}{4+x^2} dx.$$

To integrate this last integral we first use long division to simplify:

$$\int \tan^{-1}\frac{x}{2} dx = x\tan^{-1}\frac{x}{2} - \ln(4+x^2) + C$$

5. $\int \frac{x^2}{\sqrt{2+3x}}\,dx$

Method 1: Use *u*-substitution. Let $u = 2+3x$, then $\frac{1}{3}du = dx$, and $x^2 = \left(\frac{u-2}{3}\right)^2 = \frac{u^2-4u+4}{9}$. Upon making this substitution, the integral becomes

$$\int \frac{x^2}{\sqrt{2+3x}}\,dx = \frac{1}{3}\int \frac{u^2-4u+4}{9\sqrt{u}}\,du = \frac{1}{27}\int \left(u^{3/2} - 4u^{1/2} + 4u^{-1/2}\right)du$$

$$= \frac{1}{27}\left[\frac{2}{5}(2+3x)^{5/2} - \frac{8}{3}(2+3x)^{3/2} + 8(2+3x)^{1/2}\right] + C$$

Method 2: First, use integration by parts. Let $u = x^2$ and $dv = \frac{1}{\sqrt{2+3x}}dx$, then $du = 2x\,dx$ and $v = \frac{2}{3}\sqrt{2+3x}$. This gives

$$\int \frac{x^2}{\sqrt{2+3x}}\,dx = \frac{2}{3}x^2\sqrt{2+3x} - \frac{4}{3}\int x\sqrt{2+3x}\,dx$$

Then, use integration by parts again or use a *u*-substitution.

6. $\int (\ln x)^2\,dx$

Use integration by parts. Let $u = \ln x$ and $dv = \ln x\,dx$, then $du = \frac{1}{x}dx$ and $v = x\ln x - x$. The integral then becomes

$$\int (\ln x)^2 dx = x(\ln x)^2 - x\ln x - \int (\ln x - 1)dx$$
$$= x(\ln x)^2 - x\ln x - x\ln x + 2x + C = x(\ln x)^2 - 2x\ln x + 2x + C$$

7. $\int \frac{dx}{x^2+2x+5}$

Complete the square in the denominator to get $\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+4}$.

Now, divide numerator and denominator both by 4.

$$\int \frac{dx}{(x+1)^2+4} = \frac{1}{4}\int \frac{dx}{\left(\frac{x+1}{2}\right)^2+1}.$$

Use a *u*-substitution. Let $u = \frac{x+1}{2}$, then $2du = dx$. We then have

$$\frac{1}{4}\int \frac{dx}{\left(\frac{x+1}{2}\right)^2+1} = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1} u$$

Upon placing x back in, we get

$$\int \frac{dx}{x^2+2x+5} = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

8. $\int \cot x\left[\ln(\sin x)\right]dx$

First, use a *u*-substitution with $u = \ln(\sin x)$ so that $du = \frac{\cos x}{\sin x}dx = \cot x dx$.

Then, we have $\int \cot x\left[\ln(\sin x)\right]dx = \int u du = \frac{u^2}{2} + C$.

Therefore, $\int \cot x\left[\ln(\sin x)\right]dx = \frac{\left[\ln(\sin x)\right]^2}{2} + C$

9. $\int \frac{5x^2+20x+6}{x(x+1)^2}dx$ Using partial fraction decomposition, we have

$\frac{5x^2+20x+6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$. This gives $A = 6, B = -1,$ and $C = 9$.

The integral then becomes $\int \frac{5x^2+20x+6}{x(x+1)^2}dx = \int \left[\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right]dx$. Finally, we have

$$\int \frac{5x^2+20x+6}{x(x+1)^2}dx = 6\ln|x| - \ln|x+1| - \frac{9}{x+1} + C = \ln\left|\frac{x^6}{x+1}\right| - \frac{9}{x+1} + C$$

10. $\int_0^3 \frac{x^3}{\sqrt{x^2+9}}dx$

Method 1: Use a u-substitution with $u = x^2 + 9$. Then, $du = 2xdx$, so that $\frac{1}{2}du = xdx$. After making this substitution, we have

$$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}dx = \frac{1}{2}\int_9^{18} \frac{u-9}{\sqrt{u}}du = \frac{1}{2}\int_9^{18}\left(u^{1/2} - 9u^{-1/2}\right)du.$$

Integrating this last integral directly, we have

$$\frac{1}{2}\int_9^{18}\left(u^{1/2} - 9u^{-1/2}\right)du = \frac{1}{2}\left[\frac{2}{3}u^{3/2} - 18u^{1/2}\right]_9^{18} = \left[\frac{1}{3}u^{3/2} - 9u^{1/2}\right]_9^{18}$$

Evaluating, we have

$$\int_0^3 \frac{x^3}{\sqrt{x^2+1}}dx = \left[\frac{1}{3}u^{3/2} - 9u^{1/2}\right]_9^{18} = \frac{18^{3/2}}{3} - 9\cdot 18^{1/2} - \left(\frac{9^{3/2}}{3} - 9\cdot 9^{1/2}\right)$$
$$= 9\cdot 2^{3/2} - 27\cdot 2^{1/2} - (9-27) = 9\cdot 2^{3/2} - 27\cdot 2^{1/2} + 18 = 18 - 9\sqrt{2}$$

Method 2: Trigonometric substitution with $x = 3\tan\theta$ and $dx = 3\sec^2\theta d\theta$. Noting that, as *x* goes from 0 to 3, θ goes from 0 to $\frac{\pi}{4}$, the integral becomes

$$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}dx = \int_0^{\pi/4} \frac{27\tan^3\theta}{\sqrt{9\tan^2\theta+9}}3\sec^2\theta d\theta = \int_0^{\pi/4}\frac{27\tan^3\theta}{3\sec\theta}3\sec^2\theta d\theta$$
$$= 27\int_0^{\pi/4}\tan^3\theta\sec\theta d\theta = 27\int_0^{\pi/4}\tan^2\theta\tan\theta\sec\theta d\theta = 27\int_0^{\pi/4}\left(\sec^2\theta - 1\right)\tan\theta\sec\theta d\theta$$

Now, let $u = \sec\theta$, so that $du = \sec\theta\tan\theta d\theta$. Changing the limits of integration, we have

$$= 27\int_1^{\sqrt{2}}\left(u^2 - 1\right)du = 27\left[\frac{u^3}{3} - u\right]_1^{\sqrt{2}} = 27\left[\frac{\sqrt{2}^3}{3} - \sqrt{2} - \frac{1}{3} + 1\right] = 18 - 9\sqrt{2}$$

11. $\int \frac{\sqrt{4x^2+9}}{x^4}\,dx$

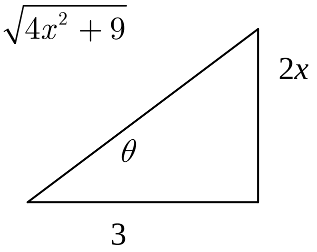
Use trigonometric-substitution with $4x^2 = 9\tan^2\theta$ or $x = \frac{3}{2}\tan\theta$. It follows that $dx = \frac{3}{2}\sec^2\theta d\theta$. Using this substitution, we have

$$\int_0^3 \frac{\sqrt{4x^2+9}}{x^4}\,dx = \int \frac{\sqrt{9\tan^2\theta+9}}{\left(\frac{3}{2}\right)^4\tan^4\theta}\frac{3}{2}\sec^2\theta d\theta = \frac{8}{9}\int \frac{\sec^3\theta}{\tan^4\theta}d\theta$$

$$= \frac{8}{9}\int \frac{\cos\theta}{\sin^4\theta}d\theta$$ Now, let $u = \sin\theta$.

$$= \frac{8}{9}\int \frac{1}{u^4}du = -\frac{8}{27u^3} = -\frac{8}{27\sin^3\theta}$$

$$= -\frac{8}{27\left(\frac{2x}{\sqrt{4x^2+9}}\right)^3} + C = -\frac{\left(4x^2+9\right)^{3/2}}{27x^3} + C$$



$$\sin\theta = \frac{2x}{\sqrt{4x^2+9}}$$

12. $\int_1^{\infty}(1-x)e^{-x}dx$

First, split the given integral into two improper integrals:

$$\int_1^{\infty}(1-x)e^{-x}dx=\int_1^{\infty}e^{-x}dx-\int_1^{\infty}xe^{-x}dx$$

Evaluate the first integral using the u-substitution $u=-x$ and $du=-dx$

$$\lim_{b\to\infty}\int_1^b e^{-x}dx=\lim_{b\to\infty}\left[-e^{-x}\right]_1^b=\lim_{b\to\infty}\left(-e^{-b}+e^{-1}\right)=\frac{1}{e}$$

Evaluate the second integral using integration by parts with $u=x$ and $dv=e^{-x}dx$ so that $du=dx$ and $v=-e^{-x}$

$$\int_1^{\infty}xe^{-x}dx=\lim_{b\to\infty}\int_1^b xe^{-x}dx=\lim_{b\to\infty}\left[-xe^{-x}\right]_1^b+\lim_{b\to\infty}\int_1^b e^{-x}dx$$

$$=\lim_{b\to\infty}\left[-be^{-b}+\frac{1}{e}\right]+\lim_{b\to\infty}\left[-e^{-x}\right]_1^b=\frac{1}{e}+\lim_{b\to\infty}\left[-e^{-b}+\frac{1}{e}\right]_1^b=\frac{2}{e}$$

We use L’Hopital’s rule to find the limit:

$$\lim_{b\to\infty}\left[-be^{-b}\right]=\lim_{b\to\infty}\left[\frac{-b}{e^b}\right]=\lim_{b\to\infty}\left[\frac{-1}{e^b}\right]=0$$

Therefore, $\int_1^{\infty}(1-x)e^{-x}dx=\int_1^{\infty}e^{-x}dx-\int_1^{\infty}xe^{-x}dx=\frac{1}{e}-\frac{2}{e}=-\frac{1}{e}$

13. $\int_0^{\pi/3}\frac{\sin^3 xdx}{\sqrt{\cos x}}$

Using a *u*-substitution, let $u=\cos x$, then $du=-\sin xdx$. After making this substitution, we have

$$\int_0^{\pi/3}\frac{\sin^3 xdx}{\sqrt{\cos x}}=\int_0^{\pi/3}\frac{\sin^2 x\sin xdx}{\sqrt{\cos x}}=\int_0^{\pi/3}\frac{\left(1-\cos^2 x\right)\sin xdx}{\sqrt{\cos x}}$$

$$=-\int_1^{1/2}\frac{1-u^2}{\sqrt{u}}du=-\int_1^{1/2}\left(u^{-1/2}-u^{3/2}\right)du=\int_{1/2}^{1}\left(u^{-1/2}-u^{3/2}\right)du$$

$$=\int_{1/2}^{1}\left(u^{-1/2}-u^{3/2}\right)du=\left[2u^{1/2}-\frac{2}{5}u^{5/2}\right]_{1/2}^{1}=2-\frac{2}{5}-2\sqrt{\frac{1}{2}}+\frac{2}{5}\sqrt{\frac{1}{2}}^{5}$$

$$=\frac{8}{5}-\sqrt{2}+\frac{\sqrt{2}}{20}=\frac{8}{5}-\frac{19\sqrt{2}}{20}\approx 0.2564971157$$

14. This question has two parts:

a) Prove the reduction formula:

$$\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$$

Using integration by parts with $u = \cos^{n-1} x$ and $dv = \cos x dx$, so that $du = -(n-1)\cos^{n-2} x \sin x dx$ and $v = \sin x$, we have

$$\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x dx \quad (1)$$

The last interval on the right-hand side can be simplified using trigonometric identities.

$$\int \cos^{n-2} x \sin^2 x dx = \int \cos^{n-2} x \left(1 - \cos^2 x\right) dx = \int \cos^{n-2} x dx - \int \cos^n x dx$$

Putting this expression back into (1) in place of the last integral, we have

$$\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\left(\int \cos^{n-2} x dx - \int \cos^n x dx\right)$$

$$\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x dx$$

Adding the integral on the far right hand side to both sides, we have

$$n\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx .$$

Now, divide by *n* and the reduction formula has been proved.

$$\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n}\int \cos^{n-2} x dx$$

15. The work done against gravity in propelling an object with mass *m* kg to an altitude of *h* m above the surface of the earth is given by

$$W = \int_{6.37\text{x}10^6}^{6.37\text{x}10^6+h} \frac{GM_E m}{r^2}\,dr$$

where $6.37\text{x}10^6$ m is the radius of the earth, $G \approx 6.667\times10^{-11}\,\text{N}\cdot m^2/kg^2$, and $M_E \approx 5.90\times10^{24}\,kg$ is the mass of the earth.

a) Find the work required to launch a 1000-kg satellite vertically to an altitude of 1000 km.

b) The formula shows that the work is dependent on *h*. Show that $\int_{6.37\text{x}10^6}^{\infty} \frac{GM_E m}{r^2}\,dr$ is convergent.

Solution: a)

I first find the antiderivative and then evaluate the expression for the given values.

$$\int \frac{GM_E m}{r^2}\,dr = GM_E m \int \frac{1}{r^2}\,dr = \frac{-GM_E m}{r}$$

It follows that $W = \int_{6.37\text{x}10^6}^{6.37\text{x}10^6+h} \frac{GM_E m}{r^2}\,dr$

$$= -\frac{GM_E m}{r}\Bigg|_{6.37\text{x}10^6}^{6.37\text{x}10^6+h} = GM_E m\left(\frac{1}{6.37\text{x}10^6} - \frac{1}{6.37\text{x}10^6+h}\right)$$

Substituting in the given values, we have

$$\left(6.667\times10^{-11}\,\text{N}\cdot m^2/kg^2\right)\left(5.90\times10^{24}\,kg\right)(1000kg)\left(\frac{1}{6.37\text{x}10^6} - \frac{1}{6.37\text{x}10^6+1000}\right)$$

$$\approx 9692491.513 units$$

Solution: b)

$$\int_{6.37\text{x}10^6}^{\infty} \frac{GM_E m}{r^2}\,dr = GM_E m \lim_{b\to\infty} \int_{6.37\text{x}10^6}^{b} \frac{1}{r^2}dr$$

$$= GM_E m \lim_{b\to\infty}\left(-\frac{1}{r}\right)_{6.37\times10^6}^{b} = GM_E m \lim_{b\to\infty}\left(\frac{1}{6.37\times10^6} - \frac{1}{b}\right)$$

$= GM_E m\left(\frac{1}{6.37\times10^6}\right)$. Therefore, the integral is convergent.

16. A string stretched between two points (0,0) and (0,2) is plucked by displacing the string *h* units at its midpoint. The motion of the string is modeled by the Fourier Sine Series whose coefficients are given by

$$b_n = h\int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx + h\int_1^2 (-x+2)\sin\left(\frac{n\pi x}{2}\right)dx\,, \tag{1}$$

where *n* is an integer. Evaluate b_n.

Solution: We handle the two integrals separately. First, consider $h\int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx$. Using integration by parts with $u = x$ and $dv = \sin\left(\frac{n\pi x}{2}\right)dx$, so that $du = dx$ and $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$, we have

$$h\int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx = \left[-\frac{2hx}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_0^1 + \frac{2h}{n\pi}\int_0^1 \cos\left(\frac{n\pi x}{2}\right)$$

$$= \left[-\frac{2h}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right] + \left[\frac{4h}{(n\pi)^2}\sin\left(\frac{n\pi x}{2}\right)\right]_0^1$$

$$= \left[-\frac{2h}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right] + \left[\frac{4h}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

Next, we consider $h\int_1^2 (-x+2)\sin\left(\frac{n\pi x}{2}\right)dx$. First, note that we can write the given integral as two integrals.

$$h\int_1^2 (-x+2)\sin\left(\frac{n\pi x}{2}\right)dx = -h\int_1^2 x\sin\left(\frac{n\pi x}{2}\right)dx + 2h\int_1^2 \sin\left(\frac{n\pi x}{2}\right)dx \tag{2}$$

The first integral we evaluate using integration by parts as before,

$$-h\int_1^2 x\sin\left(\frac{n\pi x}{2}\right)dx = \left[\frac{2hx}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_1^2 - \frac{2h}{n\pi}\int_1^2 \cos\left(\frac{n\pi x}{2}\right)$$

$$= \left[\frac{4h}{n\pi}\cos(n\pi) - \frac{2h}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right] - \left[\frac{4h}{(n\pi)^2}\sin\left(\frac{n\pi x}{2}\right)\right]_1^2$$

$$= \left[\frac{4h}{n\pi}\cos(n\pi) - \frac{2h}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right] - \left[\frac{4h}{(n\pi)^2}\sin(n\pi) - \frac{4h}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{4h}{n\pi}\cos(n\pi) - \frac{2h}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4h}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

The 2nd integral on the right-hand side of (2) can be done directly (or with a *u* substitution):

$$2h\int_1^2 \sin\left(\frac{n\pi x}{2}\right)dx = -\frac{4h}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_1^2 = -\frac{4h}{n\pi}\cos(n\pi) + \frac{4h}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

Putting these results together, we see that (1) is given by

$$b_n = h\int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx + h\int_1^2 (-x+2)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= -\frac{2h}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4h}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) + \frac{4h}{n\pi}\cos(n\pi) - \frac{2h}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4h}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) - \frac{4h}{n\pi}\cos(n\pi) + \frac{4h}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

It follows that $b_n = \frac{8h}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$.

17. Use (a) Simpson's Rule (b) the Trapezoidal Rule, and (c) the Midpoint Rule with $n = 6$ to approximate $\frac{1}{\sqrt{2\pi}}\int_{-3}^{3} e^{-x^2/2}\,dx$. Find the maximum possible error using each approach.

Solution:

a) 0.9972474675
b) 0.9952975109
c) 0.9982224458

18. If the area under the graph of $y = \frac{1}{1+x^2}$ is revolved around the *x*-axis the area of the surface generated is given by the integral $A = 2\pi\int_0^1 \frac{x\sqrt{\left(1+x^2\right)^4 + \left(1-x^2\right)^2}}{\left(1+x^2\right)^3}\,dx$.

Approximate the surface area to three decimal places.

Answer: $A \approx 2.383150404$

19. In a chemical reaction, one unit of compound *Y* and one unit of compound *Z* are converted into a single unit of compound *X*. If the amount of compound *X* formed and the rate of formation of *X* is proportional to the amount of unconverted compounds *Y* and *Z*, then

$$\frac{dx}{dt} = k(y_0 - x)(z_0 - x)$$

where y_0 and z_0 are the initial amounts of substances *y* and *z*, respectively. From the above equations we obtain,

$$\int \frac{1}{(y_0 - x)(z_0 - x)} dx = \int k dt \tag{1}$$

Use partial fraction decomposition to integrate the left-hand side and then solve for *x* in terms of *t*.

Using partial fraction decomposition, we assume that the integrand decomposes into fractions of the form: $\frac{1}{(y_0 - x)(z_0 - x)} = \frac{A}{y_0 - x} + \frac{B}{z_0 - x}$, this gives us the equation

$$1 = A(z_0 - x) + B(y_0 - x)\,.$$

Choosing $x = z_0$, we have $B = \frac{1}{y_0 - z_0}$. Choosing $x = y_0$, we have $A = \frac{1}{z_0 - y_0}$.

It follows that $\int \frac{1}{(y_0 - x)(z_0 - x)} dx = \int \frac{1}{z_0 - y_0} \frac{1}{y_0 - x} + \frac{1}{y_0 - z_0} \frac{1}{z_0 - x} dx$

$$= \frac{1}{z_0 - y_0} \int \frac{1}{y_0 - x} - \frac{1}{z_0 - x} dx$$

The last integral on the right is evaluated using *u*-substitutions.

$$\frac{1}{z_0 - y_0}\int \frac{1}{y_0 - x} - \frac{1}{z_0 - x}dx = \frac{1}{z_0 - y_0}\left[\ln\left|z_0 - x\right| - \ln\left|y_0 - x\right|\right]$$

$$= \frac{1}{z_0 - y_0}\ln\left|\frac{z_0 - x}{y_0 - x}\right|$$

The right-hand side of (1) gives $kt + C$.
Equating the two sides of (1) we get

$$\frac{1}{z_0 - y_0}\ln\left|\frac{z_0 - x}{y_0 - x}\right| = kt + C_1.$$

It remains to solve for *x* as a function of *t*. Multiplying both sides by $z_0 - y_0$, we get

$$\ln\left|\frac{z_0 - x}{y_0 - x}\right| = \left(z_0 - y_0\right)kt + C_2$$

Rewriting in exponential form, we have

$$\left|\frac{z_0 - x}{y_0 - x}\right| = e^{(z_0 - y_0)kt + C_2} = ce^{(z_0 - y_0)kt}$$

$$\frac{z_0 - x}{y_0 - x} = ce^{(z_0 - y_0)kt}$$

$$z_0 - x = \left(y_0 - x\right)ce^{(z_0 - y_0)kt}$$

$$z_0 - x = y_0 ce^{(z_0 - y_0)kt} - xce^{(z_0 - y_0)kt}$$

We get the terms that contain *x* on the same side:

$$x - xce^{(z_0 - y_0)kt} = z_0 - y_0 ce^{(z_0 - y_0)kt}$$

Factor out the *x*:

$$x\left(1 - ce^{(z_0 - y_0)kt}\right) = z_0 - y_0 ce^{(z_0 - y_0)kt}$$

$$x = \frac{z_0 - y_0 ce^{(z_0 - y_0)kt}}{1 - ce^{(z_0 - y_0)kt}}$$

20. Use the substitution $u = \tan\left(\frac{\theta}{2}\right)$ and the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ to evaluate the integral $\int_0^{\pi/2} \frac{d\theta}{4\sin\theta + 3\cos\theta}$.

Solution: The identity leads to the formulas $\sin\theta = \frac{2u}{1+u^2}$, $\cos\theta = \frac{1-u^2}{1+u^2}$, and $d\theta = \frac{2du}{1+u^2}$. Note that *u* goes from 0 to 1 as θ goes from 0 to 2π.

$$\int_0^{\pi/2} \frac{d\theta}{4\sin\theta + 3\cos\theta} = \int_0^1 \frac{\frac{2du}{1+u^2}}{4\left(\frac{2u}{1+u^2}\right) + 3\left(\frac{1-u^2}{1+u^2}\right)}$$

$$= \int_0^1 \frac{2du}{4(2u) + 3\left(1-u^2\right)}$$

$$= \int_0^1 \frac{2}{8u + 3(1-u^2)} du$$

$$= \int_0^1 \frac{2}{(3u+1)(3-u)} du$$

Using partial fraction decomposition, the last integral becomes

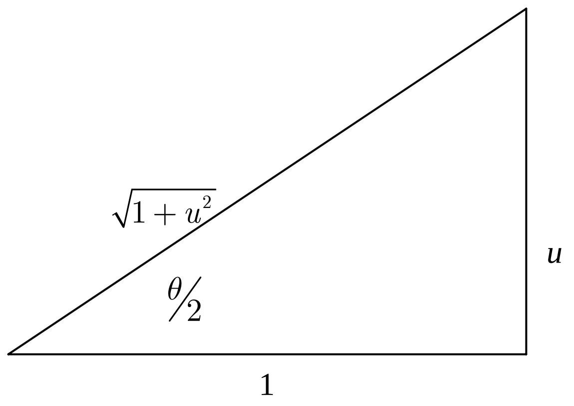
$$= \int_0^1 \left(\frac{3/5}{3u+1} + \frac{1/5}{3-u}\right) du$$

$$= \left[\frac{1}{5}\ln|3u+1| - \frac{1}{5}\ln|3-u|\right]_0^1$$

$$= \frac{1}{5}\ln|4| - \frac{1}{5}\ln|2| - \left(\frac{1}{5}\ln|1| - \frac{1}{5}\ln|3|\right)$$

$$= \frac{1}{5}(\ln 4 - \ln 2 + \ln 3) = \frac{1}{5}\ln 6$$

Why the substitution in #20 works.



From the triangle, we see that

$$\frac{u}{1} = \tan\left(\frac{\theta}{2}\right) \quad \sin\frac{\theta}{2} = \frac{u}{\sqrt{1+u^2}} \qquad \cos\frac{\theta}{2} = \frac{1}{\sqrt{1+u^2}}$$

Now use the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ in the form $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$. This gives $\cos\theta = \left(\frac{1}{\sqrt{1+u^2}}\right)^2 - \left(\frac{u}{\sqrt{1+u^2}}\right)^2 = \frac{1-u^2}{1+u^2}$. We could solve for $\sin\theta$ directly or we can also use the identity $\sin 2\theta = 2\sin\theta\cos\theta$ in the form $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$.

This gives $\sin\theta = 2\left(\frac{1}{\sqrt{1+u^2}}\right)\left(\frac{u}{\sqrt{1+u^2}}\right) = \frac{2u}{1+u^2}$

This method will work for any rational function of sine and cosine.

#5 **Method 2**: Use integration by parts twice. First let $u = x^2$ and $dv = (x+1)^{1/2}\,dx$ so that $du = 2x dx$ and $v = \frac{2}{3}(x+1)^{3/2}$. Then,

$$\int x^2\sqrt{x+1}dx = \frac{2}{3}x^2\,(x+1)^{3/2} - \frac{4}{3}\int x\,(x+1)^{3/2}dx$$

Compute the last integral using integration by parts again. Let $u = x$ and $dv = (x+1)^{3/2}\,dx$ so that $du = dx$ and $v = \frac{2}{5}(x+1)^{5/2}$. Then,

$$\int x^2\sqrt{x+1}dx = \frac{2}{3}x^2\,(x+1)^{3/2} - \frac{8}{15}x\,(x+1)^{5/2} + \frac{8}{15}\int (x+1)^{5/2}\,dx$$

$$= \frac{2}{3}x^2\,(x+1)^{3/2} - \frac{8}{15}x\,(x+1)^{5/2} + \frac{16}{105}(x+1)^{7/2} + C$$