Name ____________________________ Test 1 Calculus 2 Mike Huff

1. $$f(x) = \int_{\sin x}^{\tan x} (1 + xt^2)dt = \int_{\sin x}^{\tan x} dt + x\int_{\sin x}^{\tan x} t^2 dt$$

$$= \tan x - \sin x + x\left[\frac{\tan^3 x}{3} - \frac{\sin^3 x}{3}\right] + C$$

It follows that

$$f'(x) = \sec^2 x - \cos x + \left[\frac{\tan^3 x}{3} - \frac{\sin^3 x}{3}\right] + x\frac{d}{dx}\left[\frac{\tan^3 x}{3} - \frac{\sin^3 x}{3}\right]$$

$$f'(x) = \sec^2 x - \cos x + \frac{1}{3}\left(\tan^3 x - \sin^3 x\right) + x\tan^2 x\sec^2 x - x\sin^2 x\cos x$$

2. $\int \tan x \sec^2 x dx$

Method 1: Using *u*-substitution: Let $u = \sec x$, then $du = \sec x \tan x dx$.
Then,

$$\int \tan x \sec^2 x dx = \int u du = \frac{u^2}{2} + C = \frac{\sec^2 x}{2} + C.$$

Method 2: Using *u*-substitution: Let $u = \tan x$, then $du = \sec^2 x dx$.
Then,

$$\int \tan x \sec^2 x dx = \int u du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C.$$

These two answers are actually the same! (They differ by a constant.)

3. $$\int_0^{\frac{\pi}{2}} \sin^5 \theta \cos^5 \theta d\theta = \int_0^{\frac{\pi}{2}} \left(\sin^2 \theta\right)^2 \cos^5 \theta \sin \theta d\theta = \int_0^{\frac{\pi}{2}} \left(1 - \cos^2 \theta\right)^2 \cos^5 \theta \sin \theta d\theta$$

Using *u*-substitution: Let $u = \cos\theta$, then $du = -\sin\theta d\theta$.
Then,

$$\int_0^{\frac{\pi}{2}} \left(1 - \cos^2 \theta\right)^2 \cos^5 \theta \sin \theta d\theta = -\int_1^0 \left(1 - u^2\right)^2 u^5 du = \int_0^1 \left(1 - u^2\right)^2 u^5 du$$

$$= \int_0^1 \left(u^5 - 2u^7 + u^9\right) du = \left.\frac{u^6}{6} - \frac{u^8}{4} + \frac{u^{10}}{10}\right|_0^1 = \frac{1}{6} - \frac{1}{4} + \frac{1}{10} = \frac{1}{60}$$

4. $\int x \tan^{-1} x dx$

Use integration by parts with $u = \tan^{-1} x$ and $dv = x$ so that $du = \frac{1}{1+x^2} dx$ and $v = \frac{x^2}{2}$. Then,

$$\int x \tan^{-1} x dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx .$$

To integrate this last integral we first use long division to simplify:

$$\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$$

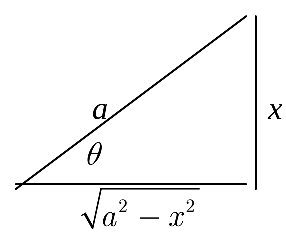
$$\int x \tan^{-1} x dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx = \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C$$

5. $\int \frac{\sqrt{a^2 - x^2}}{x^4} dx$

Use trigonometric-substitution with $x^2 = a^2 \sin^2 \theta$ or $x = a \sin \theta$. It follows that $dx = a \cos \theta d\theta$. Using this substitution, we have

$$\int \frac{\sqrt{a^2 - x^2}}{x^4} dx = \int \frac{\sqrt{a^2 - a^2 \sin^2 \theta}}{a^4 \sin^4 \theta} a \cos \theta d\theta$$

$$= \frac{1}{a^2} \int \frac{\cos^2 \theta}{\sin^4 \theta} d\theta = \frac{1}{a^2} \int \cot^2 \theta \csc^2 \theta d\theta$$



$$\cot \theta = \frac{\sqrt{a^2 - x^2}}{x}$$

Let $u = \cot \theta$, then $du = -\csc^2 \theta d\theta$.
Then, the integral becomes

$$= -\frac{1}{a^2} \int u^2 du = -\frac{u^3}{3a^2} = -\frac{\cot^3 \theta}{3a^2}$$

$$= -\frac{1}{3a^2} \left(\frac{\sqrt{a^2 - x^2}}{x}\right)^3 + C = -\frac{\left(a^2 - x^2\right)^{3/2}}{3a^2 x^3} + C$$

6. $\int x(\ln x)^2 dx$

Method 1: Use a substitution and then integration by parts.
First, use a w-substitution. Let $w = \ln x$, then $x = e^w$ and $dx = e^w dw$.
Upon making this substitution, the integral becomes

$$\int x(\ln x)^2 dx = \int w^2 e^w dw$$

Integrate the right-hand side using integration by parts with $u = w^2$ and $dv = e^w dw$. Using tabular integration by parts, we have

u	*dv*	± 1
w^2	e^w	1
$2w$	e^w	-1
2	e^w	1
	e^w	-1
		1

$$\int w^2 e^w dw = w^2 e^w - 2we^w + 2e^w + C = x(\ln x)^2 - 2x\ln x + 2x + C$$

Method 2: Use integration by parts twice on the original integral to get

$$\int x(\ln x)^2 dx = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C$$

7. $\int \frac{dx}{x^3 - 1} = \int \frac{dx}{(x-1)(x^2+x+1)}$. Using partial fraction decomposition we get

$$= \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{x+2}{x^2+x+1} \cdot \frac{2}{2} dx$$

$$= \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{6}\int \frac{2x+4}{x^2+x+1} dx$$

$$= \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{6}\int \frac{2x+1}{x^2+x+1} dx - \frac{1}{6}\int \frac{3}{x^2+x+1} dx$$

The first two integrals require u substitution in the denominator and result in log forms. To integrate the third integral we complete the square in the denominator to get $\frac{1}{2}\int \frac{dx}{x^2+x+1} = \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{2}{3}\int \frac{dx}{\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1}$.

Where we have divided numerator and denominator both by 3/4.

Use a *u*-substitution. Let $u = \frac{2x+1}{\sqrt{3}}$, then $\frac{\sqrt{3}}{2}du = dx$. We then have

$$\frac{2}{3}\int \frac{dx}{\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1} = \frac{\sqrt{3}}{3}\int \frac{du}{1+u^2} = \frac{\sqrt{3}}{3}\tan^{-1} u$$

Upon placing x back in, we get

$$\frac{1}{2}\int \frac{dx}{x^2 + x + 1} = \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

The original integral is therefore equal to

$$\frac{1}{2}\ln\left|x-1\right| - \frac{1}{6}\ln\left|x^2 + x + 1\right| - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

8. $\int \sqrt{\sqrt{x} + 1}dx$

Method 1: Using a *u*-substitution with $u^2 = \sqrt{x} + 1$, we have $x = \left(u^2 - 1\right)^2$ and $dx = 4u\left(u^2 - 1\right)du = \left(4u^3 - 4u\right)du$.

This substitution gives

$$\int \sqrt{\sqrt{x} + 1}dx = \int \sqrt{u^2}\left(4u^3 - 4u\right)du = 4\int \left(u^4 - u^2\right)du$$

$$= \frac{4}{5}u^5 - \frac{4}{3}u^3 + C$$

$$= \frac{4}{5}(\sqrt{x} + 1)^{5/2} - \frac{4}{3}(\sqrt{x} + 1)^{3/2} + C$$

9. $\int_0^1 \frac{x^3}{\sqrt{x^2+1}}dx$

Use a u-substitution with $u = x^2 + 1$. Then, $du = 2xdx$, so that $\frac{1}{2}du = xdx$. After making this substitution, we have

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}dx = \frac{1}{2}\int_1^2 \frac{u-1}{\sqrt{u}}du = \frac{1}{2}\int_1^2 \left(u^{1/2} - u^{-1/2}\right)du.$$

Integrating this last integral directly, we have

$$\frac{1}{2}\int_1^2 \left(u^{1/2} - u^{-1/2}\right)du = \frac{1}{2}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^2 = \left[\frac{1}{3}u^{3/2} - u^{1/2}\right]_1^2$$

Evaluating, we have

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}dx = \left[\frac{1}{3}u^{3/2} - u^{1/2}\right]_1^2 = \frac{2^{3/2}}{3} - 2^{1/2} - \frac{1}{3} + 1 = \frac{2-\sqrt{2}}{3}$$

10. $\int_{\sqrt{3}}^2 \frac{\sqrt{x^2-3}}{x}dx$

Use u-substitution with $u = \sqrt{3}\sec\theta$. It follows that $du = \sqrt{3}\sec\theta\tan\theta d\theta$ and that $\theta = 0$ when $x = \sqrt{3}$ and $\theta = \frac{\pi}{6}$ when $x = 2$. Using this substitution, we have

$$\int_{\sqrt{3}}^2 \frac{\sqrt{x^2-3}}{x}dx = \int_0^{\pi/6} \frac{\sqrt{3}\tan\theta}{\sqrt{3}\sec\theta}\sqrt{3}\sec\theta\tan\theta d\theta.$$

Simplifying, we get

$$\sqrt{3}\int_0^{\pi/6} \tan^2\theta d\theta = \sqrt{3}\int_0^{\pi/6}\left(\sec^2\theta - 1\right)d\theta = \sqrt{3}[\tan\theta - \theta]_0^{\pi/6}$$

Finally, evaluating at the limits of integration, we have

$$\int_{\sqrt{3}}^2 \frac{\sqrt{x^2-3}}{x}dx = \sqrt{3}[\tan\theta - \theta]_0^{\pi/6} = \sqrt{3}\left[\tan\frac{\pi}{6} - \frac{\pi}{6}\right] = \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right) = 1 - \frac{\sqrt{3}\pi}{6}$$

11. $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$

First, split the given integral into two improper integrals:

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx = \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}} dx + \int_{0}^{\infty} \frac{e^x}{1+e^{2x}} dx = \lim_{a \to -\infty} \int_{a}^{0} \frac{e^x}{1+e^{2x}} dx + \lim_{b \to \infty} \int_{0}^{b} \frac{e^x}{1+e^{2x}} dx$$

Next, evaluate each integral separately using the u-substitution $u = e^x$ and $du = e^x dx$

$$\lim_{a \to -\infty} \int_{a}^{0} \frac{e^x}{1+e^{2x}} dx = \lim_{a \to -\infty} \left[\tan^{-1} e^x\right]_a^0 = \lim_{a \to -\infty} \left(\tan^{-1} 1 - \tan^{-1} e^a\right) = \frac{\pi}{4}$$

Similarly,

$$\lim_{b \to \infty} \int_{0}^{b} \frac{e^x}{1+e^{2x}} dx = \lim_{b \to \infty} \left[\tan^{-1} e^x\right]_0^b = \lim_{b \to \infty} \left(\tan^{-1} e^b - \tan^{-1} 1\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Therefore, $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx = \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}} dx + \int_{0}^{\infty} \frac{e^x}{1+e^{2x}} dx = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$

12. $\int_{0}^{\infty} \frac{dx}{\sqrt{x}\,(x+1)}$

Since the integrand is discontinuous at $x = 0$, we split the integral into two integrals.

$$\int_{0}^{\infty} \frac{dx}{\sqrt{x}\left(x+1\right)} = \lim_{a \to 0} \int_{a}^{1} \frac{dx}{\sqrt{x}\left(x+1\right)} + \lim_{b \to \infty} \int_{1}^{b} \frac{dx}{\sqrt{x}\left(x+1\right)}$$

The antiderivatives of both can be obtained using a *u*-substitution. Let $u = \sqrt{x}$, then $u^2 = x$, $u^2 + 1 = x + 1$, and $du = 2udu$. Making this substitution, we get

$$\int \frac{dx}{\sqrt{x}\,(x+1)} = \int \frac{1}{u\left(u^2+1\right)} 2udu = 2\tan^{-1} \sqrt{x}$$

$$\int_{0}^{\infty} \frac{dx}{\sqrt{x}\,(x+1)} = \lim_{a \to 0} 2\tan^{-1} \sqrt{x}\Big]_a^1 + \lim_{b \to \infty} 2\tan^{-1} \sqrt{x}\Big]_1^b$$

$$= \lim_{a \to 0} \left(2\tan^{-1} \sqrt{1} - 2\tan^{-1} \sqrt{a}\right) + \lim_{b \to \infty} \left(2\tan^{-1} \sqrt{b} - 2\tan^{-1} \sqrt{1}\right)$$

$$= \frac{\pi}{2} + 2\frac{\pi}{2} - \frac{\pi}{2} = \pi$$

13. We first make a *u* substitution, letting $u = e^x$ so that $du = e^x dx$. This gives

$$\int \frac{e^{4x}}{e^{2x} + 3e^x + 2} dx = \int \frac{u^3}{u^2 + 3u + 2} du .$$

Polynomial long division on the resulting improper rational integrand gives

$$\frac{u^3}{u^2 + 3u + 2} = u - 3 + \frac{7u + 6}{u^2 + 3u + 2} .$$

Thus, the original integral yields the two integrals:

$$= \int (u - 3)\, du + \int \frac{7u + 6}{u^2 + 3u + 2} du .$$

The first requires integration of a polynomial:

$$\int (u - 3)\, du = \frac{u^2}{2} - 3u + c .$$

The second requires us to use partial fraction decomposition.

$$\int \frac{7u + 6}{(u + 2)(u + 1)} du = \int \frac{8}{u + 2} du - \int \frac{1}{u + 1} du$$
$$= 8 \ln|u + 2| - \ln|u + 1|$$

Putting the pieces together, we have

$$\int \frac{u^3}{u^2 + 3u + 2} du = \frac{u^2}{2} - 3u + 8 \ln|u + 2| - \ln|u + 1| + C$$

Therefore,

$$\int \frac{e^{4x}}{e^{2x} + 3e^x + 2} dx = \frac{e^{2x}}{2} - 3e^x + 8 \ln|e^x + 2| - \ln|e^x + 1| + C$$

14. This answer has two parts:

a) Integrating by parts twice, we see that

$$\int e^{ax} \cos bx dx = e^{ax} \left(\frac{b \sin bx + a \cos bx}{a^2 + b^2} \right) + c$$

b) Evaluate $\int_0^{\pi/10} e^{3x} \cos 5x dx$

To use the formula from (a) to evaluate the integral, we let $a = 3$ and $b = 5$. Then,

$$\int_0^{\pi/10} e^{3x} \cos 5x dx = e^{3x} \left(\frac{5 \sin 5x + 3 \cos 5x}{3^2 + 5^2} \right) \Bigg|_0^{\pi/10}$$

$$= \frac{e^{3x}}{34}(5\sin 5x + 3\cos 5x)\Big|_0^{\pi/10}$$

$$= \frac{e^{3\pi/10}}{34}\left(5\sin\frac{\pi}{2} + 3\cos\frac{\pi}{2}\right) - \frac{e^0}{34}(5\sin 0 + 3\cos 0)$$

$$= \frac{e^{3\pi/10}}{34}(5+0) - \frac{3}{34} = \frac{5e^{3\pi/10} - 3}{34}$$

15. To integrate $\int_{-2}^{3}\left|1-x^2\right|dx$, we must use three integrals because of the definition of the absolute value function in the integrand:

$$\left|1-x^2\right| = \begin{cases} -\left(1-x^2\right) & x < -1 \\ 1-x^2 & -1 \le x \le 1 \\ -\left(1-x^2\right) & x > 1 \end{cases}$$

Thus,

$$\int_{-2}^{3}\left|1-x^2\right|dx = \int_{-2}^{-1}\left(x^2-1\right)dx + \int_{-1}^{1}\left(1-x^2\right)dx + \int_{1}^{3}\left(x^2-1\right)dx$$

$$= \frac{1}{3}x^3 - x\Big|_{-2}^{-1} + x - \frac{1}{3}x^3\Big|_{-1}^{1} + \frac{1}{3}x^3 - x\Big|_{1}^{3}$$

$$= \frac{1}{3}(-1)^3 - (-1) - \left(\frac{1}{3}(-2)^3 - (-2)\right) + 1 - \frac{1}{3}1^3 - (-1) + \frac{1}{3}(-1)^3 + \frac{1}{3}3^3 - 3 - \frac{1}{3} + 1$$

$$= \frac{4}{3} + \frac{4}{3} + \frac{20}{3}$$

$$= \frac{28}{3}$$

16. The Fourier series analysis of the sawtooth wave requires the computation of the integral

$$b_m = \frac{\omega^2 A}{2\pi^2}\int_{-\pi/\omega}^{\pi/\omega} t\sin(m\omega t)dt\,,$$

where m is an integer and ω and A are nonzero constants. Compute it.

Answer: $b_m = \dfrac{-A}{\pi m}\cos(m\pi)$