Name ___________________________ Test 1A Calculus 2 Mike Huff Spring 2010

1. $\int \frac{1}{\sqrt{1+e^x}} dx$

2. $\int \sec^3 x dx$

3. $\int_0^{2\pi} \sin mx \sin nx dx$, where m and n are integers with $m \neq \pm n$.

4. $\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2) dx$

5. $\int x^3 \sqrt{x+1} dx$

6. $\int \frac{1}{1+\tan^2(x)} dx$

7. $\int \frac{x+1}{x^2-2x+10} dx$

8. $\int \sin(\ln x) dx$

9. $\int_0^{\pi/4} \sqrt{1+\cos 4x} dx$

10. $\int \frac{1}{x^2\sqrt{16-x^2}} dx$

11. $\int_0^{\infty} x^4 e^{-x} dx$

12. $\int_0^2 \frac{dx}{1-x^2}$

13. **The Gamma Function**: Euler's Gamma function is defined by the formula:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \ x > 0.$$

a) Show that $\Gamma(1) = 1$.

b) Apply integration by parts to the integral for $\Gamma(x+1)$ to show that $\Gamma(x+1) = x\Gamma(x)$.

1) $\int \frac{1}{\sqrt{1+e^x}}dx$

Use a *u*-substitution with $u=\sqrt{1+e^x}$ so that $u^2=1+e^x$ and $2udu=e^x dx$. Solving this last equation for

$$dx=\frac{2u}{e^x}du=\frac{2u}{u^2-1}du$$

and substituting gives

$$\begin{aligned}\int \frac{1}{\sqrt{1+e^x}}dx &= \int \frac{2u}{u\left(u^2-1\right)}du \\ &= 2\int \frac{du}{\left(u+1\right)\left(u-1\right)} \qquad \text{partial fraction decomposition} \\ &= 2\left[\frac{1}{2}\int\left(\frac{1}{u-1}-\frac{1}{u+1}\right)du\right] \\ &= \ln\left|u-1\right|-\ln\left|u+1\right|+C \\ &= \ln\frac{\left|u-1\right|}{\left|u+1\right|}+C \\ &= \ln\frac{\left|\sqrt{1+e^x}-1\right|}{\left|\sqrt{1+e^x}+1\right|}+C\end{aligned}$$

2) $\int \sec^3 x dx$

First rewrite the integrand as $\sec^3 x=\sec^2 x\sec x$. Then use integration by parts with $u=\sec x$ and $dv=\sec^2 xdx$ so that $du=\sec x\tan xdx$ and $v=\tan x$.
Then

$$\begin{aligned}\int \sec^3 xdx &= \int \sec^2 x\sec xdx \\ &= \sec x\tan x-\int \sec x\tan^2 xdx\end{aligned}$$

Now use the trig identity $\tan^2 x=\sec^2 x-1$

$$\begin{aligned}&= \sec x\tan x-\int \sec x\left(\sec^2 x-1\right)dx \\ &= \sec x\tan x-\int\left(\sec^3 x-\sec x\right)dx \\ &= \sec x\tan x-\int \sec^3 xdx+\int \sec xdx \\ &= \sec x\tan x+\ln\left|\sec x+\tan x\right|-\int \sec^3 xdx\end{aligned}$$

Add the last integral back to both sides to get

$$2\int \sec^3 xdx=\sec x\tan x+\ln\left|\sec x+\tan x\right|$$

or

$$\int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$$

3) $\int_0^{2\pi} \sin mx \sin nx dx$, where m and n are integers with $m \neq \pm n$.

Use integration by parts with $u = \sin mx$ and $dv = \sin nx dx$ so that $du = m\cos mx dx$ and $v = -\frac{1}{n}\cos nx$.

$$\int \sin mx \sin nx dx = \sin mx\left(-\frac{1}{n}\cos nx\right) - \int -\frac{1}{n}\cos nx\left(m\cos mx\right)dx$$

$$= -\frac{1}{n}\sin mx \cos nx + \frac{m}{n}\int \cos nx \cos mx dx$$

Do integration by parts again with $u = \cos mx$ and $dv = \cos nx dx$ so that $du = -m\sin mx dx$ and $v = \frac{1}{n}\sin nx$.

$$= -\frac{1}{n}\sin mx \cos nx + \frac{m}{n}\left[\cos mx \frac{1}{n}\sin nx - \int \frac{1}{n}\sin nx\left(-m\sin mx\right)dx\right]$$

$$= -\frac{1}{n}\sin mx \cos nx + \frac{m}{n^2}\cos mx \sin nx + \frac{m^2}{n^2}\int \sin nx\left(\sin mx\right)dx$$

Subtract the last integral from both sides to get

$$\left(1 - \frac{m^2}{n^2}\right)\int \sin mx \sin nx dx = -\frac{1}{n}\sin mx \cos nx + \frac{m}{n^2}\cos mx \sin nx$$

Divide through by $1 - \frac{m^2}{n^2} = \frac{n^2 - m^2}{n^2}$ to isolate the original indefinite integral.

$$\int_0^{2\pi} \sin mx \sin nx dx = \left[\frac{n}{m^2 - n^2}\sin mx \cos nx + \frac{m}{n^2 - m^2}\cos mx \sin nx\right]_0^{2\pi}$$

$$= 0$$

The integral evaluates to zero since the sine of an integer multiple of π is 0.

4. $\int_0^{1/\sqrt{2}} 2x\sin^{-1}(x^2)dx$

Use integration by parts with $u = \sin^{-1}\left(x^2\right)$ and $dv = 2xdx$ so that $du = \dfrac{2x}{\sqrt{1-x^4}}dx$ and $v = x^2$. Then

$$\int 2x\sin^{-1}(x^2)dx = x^2\sin^{-1}\left(x^2\right) - \int \frac{2x^3}{\sqrt{1-x^4}}dx$$

The last integral can be done by u-substitution with $u = 1 - x^4$ so that $du = -4x^3dx$.

$$\int \frac{2x^3}{\sqrt{1-x^4}}dx = -\frac{1}{2}\int u^{-\frac{1}{2}}du = -\sqrt{u} = -\sqrt{1-x^4}\,.$$

Therefore,

$$\begin{aligned}\int_0^{\frac{1}{\sqrt{2}}} 2x\sin^{-1}(x^2)dx &= \left[x^2\sin^{-1}\left(x^2\right) + \sqrt{1-x^4}\,\right]_0^{\frac{1}{\sqrt{2}}} \\ &= \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) + \sqrt{1-\frac{1}{4}} - 1 \\ &= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \approx 0.12782479\end{aligned}$$

5. $\int x^3\sqrt{x+1}dx$ Use repeated integration by parts:

u	dv	± 1
x^3	$\sqrt{x+1} = (x+1)^{\frac{1}{2}}$	1
$3x^2$	$\frac{2}{3}(x+1)^{\frac{3}{2}}$	-1
$6x$	$\frac{4}{15}(x+1)^{\frac{5}{2}}$	1
6	$\frac{8}{105}(x+1)^{\frac{7}{2}}$	-1
	$\frac{16}{945}(x+1)^{\frac{9}{2}}$	1
		-1

$$\int x^3\sqrt{x+1}dx$$
$$= \frac{2}{3}x^3(x+1)^{\frac{3}{2}} - \frac{4}{5}x^2(x+1)^{\frac{5}{2}} + \frac{16}{35}x(x+1)^{\frac{7}{2}} - \frac{32}{315}(x+1)^{\frac{9}{2}} + C$$

6. Use a trig identity:

$$\begin{aligned}\int \frac{1}{1+\tan^2(x)}\,dx &= \int \frac{1}{\sec^2(x)}\,dx \\ &= \int \cos^2 x dx \\ &= \frac{1}{2}\int 1+\cos 2x dx \\ &= \frac{1}{2}x + \frac{\sin 2x}{4} + C\end{aligned}$$

7. $\displaystyle\int \frac{x+1}{x^2-2x+10}dx$

Add and subtract 1 then split the integral into two.

$$\int \frac{x+1}{x^2-2x+10}dx = \int \frac{x-1+2}{x^2-2x+10}dx$$

$$= \int \frac{x-1}{x^2-2x+10}dx + \int \frac{2}{x^2-2x+10}dx$$

The integral $\displaystyle\int \frac{x-1}{x^2-2x+10}dx$ can be done with *u*-substitution, letting $u = x^2-2x+10$ so that $du = 2x-2dx = 2(x-1)dx$.

Then

$$\int \frac{x-1}{x^2-2x+10}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln\left|x^2-2x+10\right|.$$

The other integral is done by completing the square on the denominator and then using a u-substitution.

$$\int \frac{2}{x^2-2x+10}dx = 2\int \frac{dx}{\left(x-1\right)^2+9} = \frac{2}{9}\int \frac{dx}{\left(\frac{x-1}{3}\right)^2+1}$$

$$= \frac{2}{3}\int \frac{du}{u^2+1} = \frac{2}{3}\tan^{-1}(u)$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{x-1}{3}\right)$$

Therefore, $\displaystyle\int \frac{x+1}{x^2-2x+10}dx = \frac{1}{2}\ln\left|x^2-2x+10\right| + \frac{2}{3}\tan^{-1}\left(\frac{x-1}{3}\right) + C$

8. $\int \sin(\ln x)dx$

Use a *u*-substitution with $u = \ln x$ so that $du = \frac{1}{x}dx$. It follows that $dx = xdu = e^u du$. The integral then becomes

$$\int \sin(\ln x)dx = \int e^u \sin u du .$$

This was done in class using integration by parts to get

$$\int e^u \sin u du = \frac{1}{2}\left(e^u \sin u - e^u \cos u\right).$$

$$\begin{aligned}\int \sin(\ln x)dx &= \int e^u \sin u du \\ &= \frac{1}{2}\left(e^u \sin u - e^u \cos u\right) \\ &= \frac{1}{2}\left[x\sin\left(\ln x\right) - x\cos\left(\ln x\right)\right] + C\end{aligned}$$

9. $\int_0^{\pi/4} \sqrt{1+\cos 4x}dx$

Use a trig identity: $1+\cos 4x = 2\cos^2(2x)$.
Then

$$\begin{aligned}\int_0^{\pi/4} \sqrt{1+\cos 4x}dx &= \int_0^{\pi/4} \sqrt{2\cos^2(2x)}dx \\ &= \sqrt{2}\int_0^{\pi/4} \cos(2x)dx \\ &= \left[\frac{\sqrt{2}}{2}\sin(2x)\right]_0^{\pi/4} \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

10. $\int \frac{1}{x^2\sqrt{16-x^2}}\,dx$

Use a trigonometric substitution. Let $x = 4\sin\theta$, then $x^2 = 16\sin^2\theta$ and $dx = 4\cos\theta d\theta$.

$$\int \frac{1}{x^2\sqrt{16-x^2}}\,dx = \int \frac{1}{16\sin^2\theta\sqrt{16-16\sin^2\theta}}4\cos\theta d\theta$$

$$= \int \frac{\cos\theta d\theta}{16\sin^2\theta\sqrt{1-\sin^2\theta}}$$

$$= \int \frac{\cos\theta d\theta}{16\sin^2\theta\sqrt{\cos^2\theta}}$$

$$= \int \frac{d\theta}{16\sin^2\theta} = \frac{1}{16}\int \csc^2\theta d\theta$$

$$= \frac{1}{16}\left[-\cot\theta\right] + C$$
$$= -\frac{\sqrt{16-x^2}}{16x} + C$$

11. $\int_0^\infty x^4 e^{-x}dx$

Use repeated integration by parts.

u	dv	±1
x^4	e^{-x}	1
$4x^3$	$-e^{-x}$	-1
$12x^2$	e^{-x}	1
$24x$	$-e^{-x}$	-1
24	e^{-x}	1
	$-e^{-x}$	-1
		1

$$\int x^4e^{-x}dx = -e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} - 24e^{-x}$$

$$= -e^{-x}\left(x^4 + 4x^3 + 12x^2 + 24x + 24\right)$$

It follows that

$$\int_0^\infty x^4e^{-x}dx = \lim_{b\to\infty}\left[-e^{-x}\left(x^4 + 4x^3 + 12x^2 + 24x + 24\right)\right]_0^b$$
$$= \lim_{b\to\infty}\frac{-\left(b^4 + 4b^3 + 12b^2 + 24b + 24\right)}{e^b} + 24$$
$$= 24$$

Taking the limit involves using l'Hopital's rule 4 times.

12. $\int_0^2 \frac{dx}{1-x^2}$

The integral is improper so we split the integral at the point of discontinuity.

$$= \int_0^2 \frac{dx}{\left(1-x\right)\left(1+x\right)} = \int_0^1 \frac{dx}{\left(1-x\right)\left(1+x\right)} + \int_1^2 \frac{dx}{\left(1-x\right)\left(1+x\right)}$$

The first integral can be done by partial fraction decomposition

$$\int_0^1 \frac{dx}{\left(1-x\right)\left(1+x\right)} = \lim_{b\to1^-}\frac{1}{2}\int_0^b \frac{1}{1-x} + \frac{1}{1+x}dx$$
$$= \lim_{b\to1^-}\frac{1}{2}\left[-\ln\left|1-x\right| + \ln\left|1+x\right|\right]_0^b$$
$$= \frac{1}{2}\lim_{b\to1^-}\ln\frac{\left|1+b\right|}{\left|1-b\right|} = -\infty$$

Since one of the two integral diverges, the original integral diverges.

13. a) $\Gamma(1) = \int_0^\infty t^{1-1}e^{-t}dt = \int_0^\infty e^{-t}dt = \lim_{b\to\infty}\left[-e^{-t}\right]_0^b = \lim_{b\to\infty}\left(-e^{-b} + 1\right) = 1$

b) $\Gamma(x+1) = \int_0^\infty t^{(x+1)-1}e^{-t}dt = \int_0^\infty t^x e^{-t}dt$

Integration by parts with $u = t^x$ and $dv = e^{-t}dt$ so that $du = xt^{x-1}dt$ and $v = -e^t$ yields

$$\int_0^\infty t^x e^{-t}dt = \lim_{b\to\infty}\left[-t^x e^{-t}\right]\Big|_0^b + x\int_0^\infty t^{x-1}e^{-t}dt$$

$$= x\int_0^\infty t^{x-1}e^{-t}dt$$

The limit involves using LH's Rule!!

Therefore, $\Gamma(x+1) = x\int_0^\infty t^{x-1}e^{-t}dt = x\Gamma(x)$.