Section 1: Sequences

In this section, we examine sets of numbers called sequences. By evaluating a function $f(x)$ at successive positive integer values, we generate an ordered collection of real numbers. That is, we evaluate the function $f(x)$ at $n = 1, 2, 3, 4, \ldots$ to get $a_1 = f(1)$, $a_2 = f(2)$, $a_3 = f(3)$, etc. to generate the set of numbers $\{a_1, a_2, a_3, \ldots a_n, a_{n+1}, \ldots\}$. For example, the $f(x) = \frac{1}{x}$ generates the sequence $\{a_n\} = \frac{1}{n}$ which has terms $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}$. When studying sequences the properties of the underlying function $f(x)$ will often be of use in determining the behavior of the related sequence.

Definition: A **sequence** $\{a_n\}$ of real numbers $a_1, a_2, a_3, \cdots, a_n$, is a function that assigns to each positive integer *n* a real number a_n. The number a_n is called the ***nth* term** of the sequence.

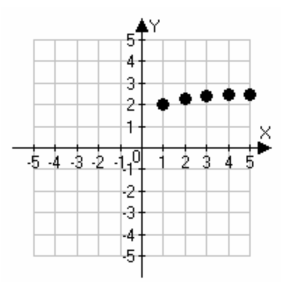
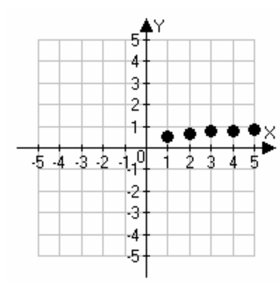
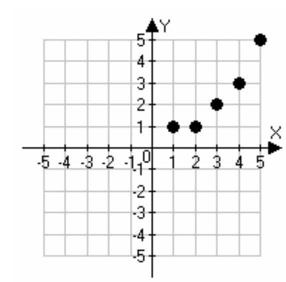
Example 1: Finding the Terms of a Sequence

Find the first five terms of each sequence and sketch a graph.

a) $a_n = \left(1 + \frac{1}{n}\right)^n$

b) $a_n = \frac{n}{n+1}$

c) The Fibonacci sequence is defined recursively as $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$

a)	b)	c)
		

Arithmetic and Geometric Sequences

Definition: An **arithmetic sequence** is a sequence in which each term is found by adding a fixed number, *d*, called the **common difference**, to the previous term.

$$a_n = a_{n-1} + d$$

In this case, we can find a formula for the nth term in terms of the first term and the common difference:

$$a_2 = a_1 + d$$
$$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d$$
$$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d$$
$$a_5 = a_4 + d = a_1 + 4d$$

In general, we have $a_n = a_1 + (n-1)d$.

Example 2: Arithmetic Sequences

Finding terms of an arithmetic sequence.

Find the 19th term of the arithmetic sequence $2, 5, 8, 11, \ldots$

Definition: A **geometric sequence** is a sequence in which each term is found by multiplying the previous term by a fixed number, *r*, called the **common ratio**

$$a_n = ra_{n-1}$$

In this case, we can find a formula for the nth term in terms of the first term and the common ratio:

$$a_2 = ra_1$$
$$a_3 = ra_2 + d = r(ra_1) = r^2 a_1$$
$$a_4 = ra_3 = r(r^2 a_1) = r^3 a_1$$

In general, we have $a_n = r^{n-1} a_1$.

Example 3: Geometric Sequences

a) Find r for the geometric sequence $8, 16, 32, \ldots, 4\left(2^{n}\right)$

b) Find the 7th term of the geometric sequence $7, 28, 112, 448, \ldots$

Limits of Sequences

If, as n gets larger, a_n approaches a number L, then L is called the **limit of the sequence**. If the sequence $a_1, a_2, a_3, \cdots, a_n$ has a limit L, we write $\lim\limits_{n\to\infty} a_n = L$.

Example 4: Sequences with Limits

Sequences with Limits

a) $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n} = e$

b) $\lim\limits_{n\to\infty}\dfrac{n}{n+1} = 1$ To see this notice that $\dfrac{n}{n+1} = 1 - \dfrac{1}{n+1}$

Convergent and Divergent Sequences

A sequence that has a limit is said to **converge** or be convergent. A sequence that does not converge is said to **diverge** or to be divergent.

Example 5: Determining Convergence or Divergence of Sequences

Do the following converge or diverge? If they converge, what is the limit?

a) $a_n = \dfrac{n+1}{n}$

b) $b_n = \dfrac{e^n}{n}$

c) $c_n = \dfrac{n}{e^n}$

The Sequences r^n ***and*** $1/n^r$

Theorem: If r is a number in the open interval $(-1,1)$ then $\lim_{n\to\infty} r^n = 0$.

Theorem: If $r > 0$, then $\lim_{n\to\infty} \dfrac{1}{n^r} = 0$.

Limit Laws for Convergent Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

a) $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

b) $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$

c) $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$

d) $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$

e) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$

f) $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ if $p > 0$ and $a_n > 0$

g) The **Squeeze Theorem for Sequences**: If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = L$ and there exists an integer N such that $a_n \le c_n \le b_n$ for all $n > N$, then $\lim_{n\to\infty} c_n = L$.

This says that the values of the middle sequence are trapped by the values of the other sequences after some point.

h) **Absolute Value Theorem**: For the sequence $\{a_n\}$, if $\lim_{n\to\infty}|a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

If this is true, then the sequence is said to converge absolutely.

i) **Theorem**: Suppose $\{a_n\} = \{f(n)\}$ for some differentiable function $f(x)$. If $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty}\{a_n\} = L$. Note: This is particularly useful in situations where L'Hopital's Rule applies.

Example 6: Determining Convergence or Divergence of Sequences

Determine the convergence or divergence of the following sequences.

a) $\{a_n\} = \{2 + (-1)^n\}$

b) $\{b_n\} = \left\{\frac{n}{1-2n}\right\}$

c) $\{c_n\} = \left\{\frac{n^2}{2n-1}\right\}$

d) $\{d_n\} = \left\{(-1)^n \frac{1}{n!}\right\}$ Hint: Use the Squeeze Theorem with $a_n = \frac{-1}{2^n}$ and $b_n = \frac{1}{2^n}$

e) Determine the *n*th term for the sequence whose first five terms are $-\frac{2}{1}, \frac{8}{2}, -\frac{26}{6}, \frac{80}{24}, -\frac{242}{120}, \ldots$ and determine if the sequence converges or diverges.

Definition: A sequence $\{a_n\}$ is **monotonic** if its terms are nondecreasing

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots$$

or if its terms are nonincreasing $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$.

Example 7: Determining if a Sequence is Monotonic

Determine whether the given sequence is monotonic.

a) $\{a_n\} = \{2 + (-1)^n\}$

b) $\{b_n\} = \left\{\frac{2n}{1+n}\right\}$

c) $\{c_n\} = \left\{\frac{n^2}{2n-1}\right\}$

d) $\{d_n\} = \left\{\frac{\cos n\pi}{n}\right\}$ Does this sequence converge absolutely?

Note that $|d_n| = \left|\frac{\cos n\pi}{n}\right| = \frac{|\cos n\pi|}{n} \leq \frac{1}{n}$, that is, $0 \leq |d_n| \leq \frac{1}{n}$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, it follows from the squeeze theorem that $\lim_{n\to\infty} |d_n| = 0$ and then from the absolute value theorem that $\lim_{n\to\infty} d_n = 0$.

Definition: A sequence $\{a_n\}$ is **bounded** if there is a positive real number *M* such that $|a_n| \le M$ for all *n*. We call *M* an upper bound of the sequence.

Example 8: Determining if a Sequence is Bounded

Determine whether each sequence is bounded.

a) $\{a_n\} = \{2 + (-1)^n\}$

b) $\{b_n\} = \left\{\dfrac{2n}{1+n}\right\}$

c) $\{c_n\} = \left\{\dfrac{n^2}{2n-1}\right\}$

Theorem: If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

Example 9: Determining if a given sequence is bounded or monotonic

Determine whether the given sequence is bounded or monotonic or both. Determine convergence if possible.

a) $\{a_n\} = \left\{\frac{1}{n}\right\}$

b) $\{b_n\} = \left\{\frac{n^2}{n+1}\right\}$

$$\left(\frac{x^2}{x+1}\right)' = \frac{(x+1)2x - x^2}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} > 0 \text{ if } x > 0$$

c) $\{c_n\} = \{(-1)^n\}$