Section 2: An Introduction to Series

Definition: Let $a_1, a_2, a_3, \cdots, a_n$ be a sequence. From this sequence a new sequence $s_1, s_2, s_3, \cdots, s_n$ can be formed:

$$
\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i
\end{aligned}
$$

The sequence of sums $s_1, s_2, s_3, \cdots, s_n$ is called the **series** obtained from the sequence $a_1, a_2, a_3, \cdots, a_n$. The sum $s_n = \sum_{i=1}^{n} a_i$ is called the **partial sum** or **nth partial sum**. If the sequence of partial sums of a series converges to *L*, then *L* is the sum of the series, that is, if $\lim_{n\to\infty} s_n = L$, then $\sum_{n=1}^{\infty} a_n = L$

For example, the sequence $\left\{\frac{1}{n}\right\}$ can be used to create the series:

$$
\begin{aligned}
s_1 &= a_1 = 1 \\
s_2 &= a_1 + a_2 = 1 + \frac{1}{2} = \frac{3}{2} \\
s_3 &= a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \\
s_4 &= s_3 + a_4 = \frac{11}{6} + \frac{1}{4} = \frac{25}{12} \\
&\vdots
\end{aligned}
$$

Geometric Series

Let *a* and *r* be real numbers. The series $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$ is called the **geometric series** with initial term *a* and ratio *r*.

Theorem: If $-1 < r < 1$, then the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \text{ converges to } \frac{a}{1-r}.$$

Proof: Let $s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1}$. Multiplying both sides of the equation by *r*, we have:

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1}$$
$$rs = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$

Subtracting the bottom equation from the top, we have

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1}$$
$$-rs = -ar - ar^2 - ar^3 - \cdots - ar^{n-1} - ar^n$$

$$s - rs = a - ar^n$$

Solving for *s*: $s = a\frac{1-r^n}{1-r}$, that is

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} = a\frac{1-r^n}{1-r}.$$

The infinite sum is found by taking the limit as $n \to \infty$. If $-1 < r < 1$, then

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} s = \lim_{n\to\infty} a\frac{1-r^n}{1-r} = \frac{a}{1-r}.$$

Example 1: Finding the sum of a geometric series

Find the sum of the series $3 + 1 + \frac{1}{3} + \cdots$, if it exists.

Example 2: Finding the sum of a geometric series

Find the sum of the series $\sum_{n=0}^{\infty} \frac{3}{2^n}$, if it exists.

Example 3: Series and decimal representations of numbers

Example 3: Series and decimal representations of numbers

Express $0.08080808\overline{08}$ as the ratio of two integers.

Example 4: Finding the sum of a geometric series

Find the sum of the series: $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$

Another type of series that will be encountered frequently is called a telescoping series because when the terms are written out the ones in the middle cancel out and the series reduces to just a few terms.

Example 5: Finding the sum of a telescoping series

Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Example 6: Finding the sum of a telescoping series

Show that the series $\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}$ is convergent and find its sum.

Properties of Infinite Series: If $\sum a_n = A$ and $\sum b_n = B$, and *k* is a real number, then the following series converge to the indicated sums:

a) $\sum_{n=1}^{\infty} ka_n = kA$

b) $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

c) $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

Example 7: Using the properties of infinite series

Find the sum of the convergent series: $\sum_{n=1}^{\infty} \left(3\left(\frac{1}{8}\right)^n - 5\left(\frac{1}{7}\right)^n \right)$

How can we quickly decide if a series diverges? We have a method that works for some series - the nth-Term Test for Divergence.

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ converges, then the sequence $\{a_n\}$ converges to 0.

Theorem: **nth-Term Test for Divergence**: If the sequence $\{a_n\}$ does not converge to 0, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 8: **Using the *n*-th Term Test for divergence**

Show that each of the following series diverges.

a) $\sum_{n=0}^{\infty} 2^n$

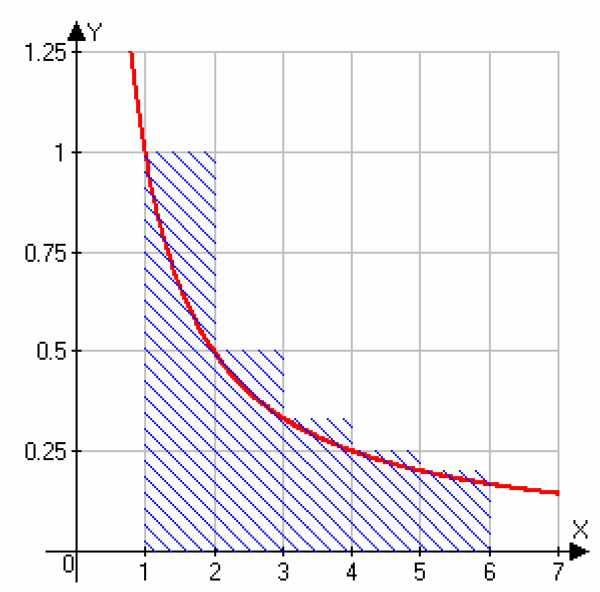
b) $\sum_{n=0}^{\infty} \frac{n!}{2n!+1}$

Example 9: The Harmonic Series

What can you say about the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{n}$?

Solution: The series diverges.

Proof: The partial sum $s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$. Below is a graph of the partial sum represented as the sum of areas of rectangles and the graph of the function $f(x) = \frac{1}{x}$.



The graph shows that each rectangle lies partly above the curve. It follows that

Sum of the areas of *n* rectangles $s_n \geq$ area under curve $= \int_1^{n+1} \frac{1}{x} dx$

$= \ln|x|_1^{n+1} = \ln(n+1)$

But $\lim_{n\to\infty} \ln|n+1| = \infty$ so $\lim_{n\to\infty} s_n = \infty$ and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Example 10: Finding the sum of a series

Find the sum of the series: $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right)$

Example 11: Finding the sum of a series

Does the series $\sum_{n=1}^{\infty}\left(\frac{2}{n^2+4n+3}\right)$ converge or diverge? If it converges, find the limit.