Section 3: Series without Negative Terms: The Comparison Test and the Integral Test

We first consider finding the sums of series that do not contain negative terms. If the terms of the sequence $\{a_n\}$ are all nonnegative, then the corresponding partial sums $s_1, s_2, s_3, \cdots, s_n$ form a non-decreasing sequence.

Definitions: A sequence $\{a_n\}$ is **bounded from above** if there is a number M such that $a_n \le M$ for all n. M is called an upper bound of the sequence. A **least upper bound** is an upper bound for a sequence that is smaller than any other upper bound.

Example 1: A Bounded Sequence

If $a_n = 1 + \frac{1}{n}$, then 2 is an upper bound for the sequence and so is any number larger than 2. No number smaller than 2 can be an upper bound, however, so 2 is the least upper bound of the sequence.

Theorem: A non-decreasing sequence that is bounded above always has a least upper bound.

Theorem: A non-decreasing sequence $\{a_n\}$ converges if and only if its terms are bounded from above, that is, there is a number M, such that $a_n \le M$ for all n.
If the sequence converges to a limit L, then $\lim_{n\to\infty} a_n = L \le M$.

Example 2: Determining the convergence of a series

The series $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ converges because all of its terms are positive and are less than or equal to the corresponding terms of $1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$.

Note: You should verify this.

The Comparison Test

One way to determine if a series converges or diverges is to compare the terms of the series with the terms of a known series. If all the terms of the series are greater than or equal to the terms of a known divergent series after a certain point in the sequence, then the series will also diverge. On the other hand, if all the terms of the series are less than or equal to the terms of a known convergent series after a fixed point in the sequence, then the series will also converge.

The Comparison Test

Let $\sum a_n$ be a series whose terms are non-negative.

a) **Test for Convergence.** The series $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \le c_n$ for all $n > N$, for some positive integer *N*.

b) **Test for Divergence.** The series $\sum a_n$ diverges if there is a divergent series of nonnegative terms $\sum d_n$ with $a_n \ge d_n$ for all $n > N$.

To use the Comparison Test, we will need to remember the results we have so far about convergent and divergent series.

Convergent Series	Divergent Series
Geometric series with $\lvert r \rvert < 1$	Geometric series with $\lvert r \rvert \ge 1$
Telescoping series like $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$	The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$
The series $\sum_{n=0}^{\infty} \frac{1}{n!}$	Any series $\sum a_n$ with $\lim_{n\to\infty} a_n \ne 0$

Example 3: Using the Comparison Test

Show that the series $\frac{2}{3}\frac{1}{1^2} + \frac{3}{4}\frac{1}{2^2} + \frac{4}{5}\frac{1}{3^2} + \cdots + \frac{(n+1)}{(n+2)}\frac{1}{n^2}\cdots = \sum_{n=1}^{\infty} \frac{(n+1)}{(n+2)}\frac{1}{n^2}$ converges by using the Comparison Test.

Note that $1 < 2$ so $n+1 < n+2$. It follows that $\frac{n+1}{n+2} < 1$ and that $\frac{(n+1)}{(n+2)}\frac{1}{n^2} < \frac{1}{n^2}$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows from the Comparison Test that $\sum_{n=1}^{\infty} \frac{(n+1)}{(n+2)}\frac{1}{n^2}$ converges.

Example 4: Using the Comparison Test

Show that the series $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ converges by using the Comparison Test.

Example 5: Using the Comparison Test

Show that the series $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$ diverges by using the Comparison Test.

Example 6: Using the Comparison Test

Show that the series $\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$ converges using the Comparison Test

Example 7: Using the Comparison Test

Show that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n - 2}}$ diverges using the Comparison Test

The Integral Test

Theorem: The Integral Test: Let $a_n = f(n)$ where $f(x)$ is continuous, positive, decreasing function for all $x \in [1, \infty)$. Then the series $\sum a_n$ and the integral $\int_1^\infty f(x)dx$ both converge or both diverge.

For our first example, we establish the convergence of the important *p*-series.

Definition: If *p* is a real constant, then the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the ***p*-series**

Example 8: Using the Integral Test

Use the integral test to show that the following is true:

The *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

a) If $p = 1$, then the series is the divergent harmonic series.

b) If $p < 1$, then the terms of the *p*-series are greater than the terms of the divergent harmonic series. Hence the series diverges, by the Comparison Test.

c) If $p > 1$, the Integral Test gives $\int_1^\infty \frac{1}{x^p} dx = \lim_{b \to \infty} \frac{x^{-p+1}}{-p+1}\Big|_1^b = \frac{1}{p-1}$ which is finite.

Hence the *p*-series converges, by the Integral Test.

Example 9: Convergence of p-series

Does the series $\sum_{n=4}^{\infty} \frac{1}{n^{1.00001}}$ converge or diverge?

Example 10: Using the Integral Test

Use the Integral Test to show that the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$.

Example 11: Using the Integral Test

Use the Integral Test to show that the series $\sum_{n=1}^{\infty} r^n$ converges for $0 < r < 1$. The Integral Test cannot be used to show the series diverges for $r > 1$. Why?

Exercises:

1. Use the Integral Test to show that the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.
2. Use the Integral Test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

The Limit Comparison Test

The Limit Comparison Test
Let $\sum a_n$ and $\sum b_n$ be series whose terms are positive for $n \geq N$ and satisfy $0 < \lim\limits_{n\to\infty} \frac{a_n}{b_n} < \infty$. Then, if $\sum b_n$ converges so does $\sum a_n$ and if $\sum b_n$ diverges so does $\sum a_n$.

Example 12: Using the Limit Comparison Test

Show that $\frac{3}{4}+\frac{5}{9}+\frac{7}{16}+\frac{9}{25}+\cdots=\sum_{n=1}^{\infty}\frac{2n+1}{(n+1)^2}$ diverges by using the Limit Comparison Test with the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$.

Solution: Let $a_n=\frac{2n+1}{(n+1)^2}$ and $b_n=\frac{1}{n}$.

Then,
$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{{}^{(2n+1)}\!\big/\!{}_{(n^2+2n+1)}}{\left({}^{1}\!/\!{}_{n}\right)}=\lim_{n\to\infty}\frac{2n^2+n}{n^2+2n+1}=2$$

Since the limit is positive and finite and $b_n=\frac{1}{n}$ diverges, by the Limit Comparison Test $\sum_{n=1}^{\infty}\frac{2n+1}{(n+1)^2}$ diverges also.

Example 13: Using the Limit Comparison Test

Show that $\sum_{n=1}^{\infty}\frac{2n}{n^2-n+1}$ diverges by using the Limit Comparison Test.

Example 14: Using the Limit Comparison Test

Show that $\sum_{n=2}^{\infty} \frac{2n^3+100n^2+1000}{(1/8)n^6-n+2}$ converges by using the Limit Comparison Test.

Exercises:

a) Does the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}$ converge or diverge

b) Does the series $\sum_{n=1}^{\infty} \frac{\sqrt{2n+1}}{n^2}$ converge or diverge?

c) Does the series $\sum_{n=1}^{\infty} \frac{n+3}{n^2\sqrt{n}}$ converge or diverge?

d) Does the series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots$ converge or diverge?

The Integral Test - Estimating the Error

Let S_n be the sum of the first *n* terms of a convergent series $\sum_{n=1}^{\infty} a_n$ whose sum is *S*. The difference $R_n = S - S_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$ is called the remainder or error in using the sum of the first *n* terms to approximate the sum of the series.

Theorem: Let $f(x)$ be a continuous, decreasing, positive function with $\int_1^{\infty} f(x)dx < \infty$.Then the error R_n in using $f(1) + f(2) + f(3) + \cdots + f(n)$ to estimate $\sum_{n=1}^{\infty} f(n)$ satisfies the inequality

$$\int_{n+1}^{\infty} f(x)dx < R_n < \int_{n}^{\infty} f(x)dx \text{, or}$$

$$S_n + \int_{n+1}^{\infty} f(x)dx < S < S_n + \int_{n}^{\infty} f(x)dx$$

Example 15: Estimating the error of an approximation

Use the first seven terms of the *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to estimate the sum of the series.

The error R_7 satisfies the inequality $\int_{7+1}^{\infty} \frac{1}{x^2} dx < R_n < \int_{7}^{\infty} \frac{1}{x^2} dx$. Thus $\frac{1}{8} < R_n < \frac{1}{7}$. The sum of the first seven terms of the series is $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} \approx 1.5118$. Hence, the sum *S* of the series satisfies the inequality: $1.5118 + \frac{1}{8} < S < 1.5118 + \frac{1}{7}$, That is, $1.6368 < S < 1.6547$.

Note: The exact value of $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.