Section 4: Alternating Series and Absolute Convergence

In the previous section, we examined series whose terms were positive numbers only. In this section, we consider those series whose terms are alternately positive or negative.

Definition: A series in which the terms are alternately positive and negative is called an **alternating series**.

Example 1: An Alternating Series

The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots+\frac{(-1)^{n+1}}{n}+\cdots$ is called the **alternating harmonic series**.

Theorem: The Alternating Series Theorem. The series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if all three of the following are satisfied:

1. $a_n > 0$ for all n.
2. $a_n \geq a_{n+1}$ for all n.
3. $a_n \to 0$.

Example 2: Using the Alternating Series Test

Show that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots+\frac{(-1)^{n+1}}{n}+\cdots$ is convergent.

Solution: It is easy to see that $a_n > 0$ for all n and $a_n \to 0$. It remains to show that $a_n \geq a_{n+1}$. Since $n+1 > n$ it follows that $\frac{1}{n} > \frac{1}{n+1}$. By the alternating series test the series converges.

Theorem: The Alternating Series Estimation Theorem. If the series alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ satisfies the three conditions above, then

$$s_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1}a_n$$

approximates the sum L of the series with an error whose absolute value is less than a_{n+1}, the value of the first unused term. Furthermore, the remainder, $L - s_n$, has the same sign as the first unused term.

Example 3: Using the alternating series estimation theorem

Find the error in using the first 100 terms to approximate $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$.

Solution: The partial sum of the first 100 terms is $s_{100} \approx 0.68817217$. The error between the sum of the series s and the partial sum s_{100} is less than a_{101}, that is $|s - s_n| < a_{n+1}$. That is, $|s - s_{100}| < a_{101} = \frac{1}{101} \approx 0.009900990$.

Absolute Convergence Test: If the series $\sum_{n=1}^{\infty}|a_n|$ converges then so does $\sum_{n=1}^{\infty}a_n$.

Definitions: Absolute and Conditional Convergence A series $\sum_{n=1}^{\infty}a_n$ is said to **converge absolutely** if $\sum_{n=1}^{\infty}|a_n|$ converges. A series $\sum_{n=1}^{\infty}a_n$ is said to **converge conditionally** if it converges but does not converge absolutely.

Example 4: Determining absolute or conditional convergence

Determine if $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n}$ converges absolutely or conditionally.

$\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n}$ is conditionally convergent since it converges but $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ is the divergent harmonic series.

The Absolute Ratio Test:

a) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

b) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

c) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test gives no information.

Example 5: Testing series for convergence

Test the following for convergence:

a) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n}$ Answer: converges absolutely.

b) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{e^n}$ Answer: diverges.

c) $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$ Answer: Ratio Test fails, use the Alternating Series Test and L'Hopital's Rule, converges.

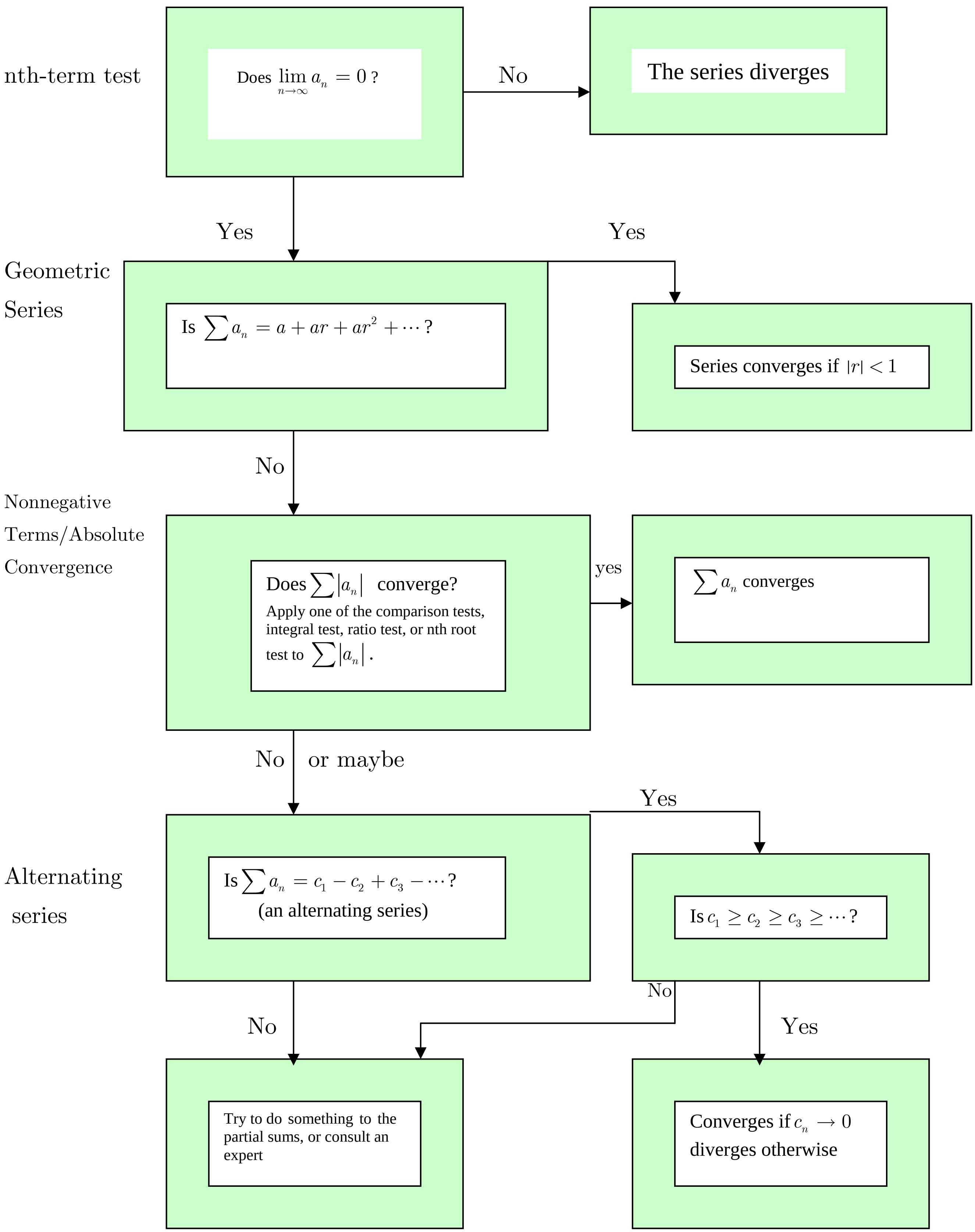
d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

e) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n!}$ Answer: converges

f) $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n}$ Answer: diverges.

g) $\sum_{n=1}^{\infty} (-1)^n \ln\left(\frac{n+2}{n}\right)$ Answer: converges use the Alternating Series Test

How to Determine Convergence of a Series



Name ______________________ Calculus 2 Mike Huff Quiz on Series

Do all problems on separate paper. Please show all work!!!

1. Determine the sum of the following convergent series:

   a) $\sum_{n=0}^{\infty}\left(\frac{2}{3^n}+\frac{1}{2^n}\right)$

   b) $\sum_{n=0}^{\infty}(-1)^n\frac{3}{2^n}$

2. Determine the convergence or divergence of the series:

   a) $\sum_{n=1}^{\infty}\frac{2n}{n^3+1}$

   b) $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$

   c) $\sum_{n=1}^{\infty}\frac{3n}{n^{3/2}+2}$

   d) $\sum_{n=1}^{\infty}\frac{\tan^{-1}n}{1+n^2}$

   e) $\sum_{n=1}^{\infty}\frac{n^4+2n-1}{n^5+3n^2+1}$

3. Determine whether the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n+2}$ converges or diverges.

4. Approximate the sum of the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$ by the 30th partial sum and estimate the error in your approximation.

5. Determine if the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n}$ is absolutely convergent

6. Determine the convergence or divergence of the series $\sum_{n=1}^{\infty}\left(\frac{3n+4}{5n-2}\right)^n$

7. Determine if each series is absolutely convergent, conditionally convergent or divergent.

   a) $\sum_{n=0}^{\infty}(-1)^n\frac{3}{n!}$

   b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n+1}$

   c) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}}{n+1}$

   d) $\sum_{n=0}^{\infty}\frac{(-2)^n(n+1)}{5^n}$

   e) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{4^n}$

   f) $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$

8. Determine the values of x for which the series $\sum_{n=0}^{\infty}(x-2)^n$ converges and find the sum.

9. Determine the values of x for which the power series $\sum_{n=0}^{\infty}\frac{nx^n}{3^{n+1}}$ converges.

10. Determine the interval and radius of convergence for the power series $\sum_{n=1}^{\infty}\frac{10^n}{n!}(x-1)^n$

11. Determine the interval and radius of convergence for the power series $\sum_{n=1}^{\infty}\frac{x^n}{n4^n}$

12. Determine the radius of convergence of the power series $\sum_{n=0}^{\infty}n!(x-5)^n$

13. Find a power series representation of $f(x)$ about $a=0$. Determine the radius of convergence.

    a) $f(x)=\frac{2}{1-x}$

    b) $f(x)=\frac{3}{1+x^2}$

    c) $f(x)=\frac{2x}{1-x^3}$

14. Determine the radius and interval of convergence and the function to which the power series converges. $\sum_{n=0}^{\infty}(3x+1)^n$

Quiz on Series: Answers

1. Determine the sum of the following convergent series:

   a) $\sum_{n=0}^{\infty}\left(\frac{2}{3^n}+\frac{1}{2^n}\right)$ Answer: 5

   b) $\sum_{n=0}^{\infty}(-1)^n\frac{3}{2^n}$ Answer: 2

2. Determine the convergence or divergence of the series:

   a) $\sum_{n=1}^{\infty}\frac{2n}{n^3+1}$ Answer: Converges

   b) $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$ Answer: Converges

   c) $\sum_{n=1}^{\infty}\frac{3n}{n^{3/2}+2}$ Answer: diverges

   d) $\sum_{n=1}^{\infty}\frac{\tan^{-1}n}{1+n^2}$ Answer: Converges

   e) $\sum_{n=1}^{\infty}\frac{n^4+2n-1}{n^5+3n^2+1}$ Answer: diverges

3. Determine whether the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n+2}$ converges or diverges.

   Answer: Diverges

4. Approximate the sum of the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$ by the 30^{th} partial sum and estimate the error in your approximation.

5. Determine if the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n}$ is absolutely convergent

   Answer: Yes is absolute values form a convergent geometric series.

6. Determine the convergence or divergence of the series $\sum_{n=1}^{\infty}\left(\frac{3n+4}{5n-2}\right)^n$

   Answer: Absolutely convergent

7. Determine if each series is absolutely convergent, conditionally convergent or divergent.

   a) $\sum_{n=0}^{\infty}(-1)^n\frac{3}{n!}$ Answer: absolutely convergent

   b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n+1}$ Answer: divergent

c) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}}{n+1}$ Answer: conditionally convergent

d) $\sum_{n=0}^{\infty}\frac{(-2)^n(n+1)}{5^n}$ Answer: absolutely convergent

e) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{4^n}$ Answer: divergent

f) $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$ Answer: conditionally convergent

8. Determine the values of x for which the series $\sum_{n=0}^{\infty}(x-2)^n$ converges and find the sum. Answer: $(1,3)$ and $\sum_{n=0}^{\infty}(x-2)^n = \frac{1}{3-x}$

9. Determine the values of x for which the power series $\sum_{n=0}^{\infty}\frac{nx^n}{3^{n+1}}$ converges.

` Answer: $(-3,3)$

10. Determine the interval and radius of convergence for the power series

$$\sum_{n=1}^{\infty}\frac{10^n}{n!}(x-1)^n$$

Answer: $(-\infty,\infty)$ and $R=\infty$

11. Determine the interval and radius of convergence for the power series $\sum_{n=1}^{\infty}\frac{x^n}{n4^n}$

Answer: $[-4,4)$ and $R=4$

12. Determine the radius of convergence of the power series $\sum_{n=0}^{\infty}n!(x-5)^n$

Answer: Converges only when $x=5$ and $R=0$

13. Find a power series representation of $f(x)$ about $a=0$. Determine the radius of convergence.

a) $f(x)=\frac{2}{1-x}$ Answer: $\sum_{n=0}^{\infty}2x^n$, $r=1$

b) $f(x)=\frac{3}{1+x^2}$ Answer: $\sum_{n=0}^{\infty}(-1)^n 3x^{2n}$, $r=1$

c) $f(x)=\frac{2x}{1-x^3}$ Answer: $\sum_{n=0}^{\infty}2x^{3n+1}$, $r=1$

14. Determine the radius and interval of convergence and the function to which the power series converges. $\sum_{n=0}^{\infty}(3x+1)^n$