Section 7 Representation of Functions as Power Series

We can now represent certain functions using power series. In this section, we make use of the fact that the function $f(x) = \frac{1}{1-x}$ has a power series representation to find power series of related functions.

Example 1: Power Series Representation of a Function

Express the function $\frac{1}{1+x}$ as a power series and find its interval of convergence.

Solution: Start with the series $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. The function $g(x) = \frac{1}{1+x}$ is related to the $f(x)$ because $g(x) = f(-x) = \frac{1}{1-(-x)}$. The power series for this function is $f(-x) = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$. Therefore,

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

Its interval of convergence is found using the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} x^{n+1}}{(-1)^n x^n}\right| = \left|\frac{x^{n+1}}{x^n}\right| = |x|$$

This converges when $|x| < 1$. To check the endpoints, evaluate at $x = -1$ and at $x = 1$. When $x = -1$, we have $\sum_{n=0}^{\infty} (-1)^n (-1)^n = \sum_{n=0}^{\infty} 1^n$ which diverges. When $x = 1$, we have $\sum_{n=0}^{\infty} (-1)^n (1)^n = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots$ which diverges.

Therefore, the interval of convergence is $-1 < x < 1$.

Example 2: Power Series Representation of a Function

Express the function $\frac{1}{2+x}$ as a power series and find its interval of convergence. Hint: $\frac{1}{2+x} = \frac{1}{2\left[1-\left(\frac{-x}{2}\right)\right]}$

Example 3: Power Series Representation of a Function

Express the function $\frac{x^2}{x+2}$ as a power series and find its interval of convergence.

Example 4: Power Series Representation of a Function

Express the function $\frac{1}{1+x^2}$ as a power series and find its interval of convergence.

Theorem: A power series can be integrated and differentiated on its radius of convergence. So, if $f(x)=\sum_{n=0}^{\infty}a_n x^n$ converges then so do $f'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}$ and $\int f(x)dx = C+\sum_{n=0}^{\infty}a_n\frac{x^{n+1}}{n+1}$.

Example 5: Power Series and their Interval of Convergence

Find a power series representation of $\tan^{-1}x$ and find its interval of convergence.

Example 6: Power Series and their Interval of Convergence

Find a power series representation of $\ln(1+x)$ and find its interval of convergence.

Example 7: Using Power Series

Evaluate the indefinite integral $\int \frac{1}{1+x^4} dx$ as a power series.

Example 8: Using Power Series

Use a power series to approximate the definite integral $\int_0^{0.5} \frac{dx}{1+x^6}$ to six decimal places.