

## Section 8 Taylor Series

Suppose the power series  $\sum_{n=0}^{\infty} a_n (x - a)^n$  has radius of convergence  $R > 0$ . As we have seen, this means that the series converges absolutely to some function  $f(x)$  on that interval.

So we have  $f(x) = \sum_{n=0}^{\infty} a_n (x - a)^n$ . Differentiating term by term, we get

$$f'(x) = \sum_{n=1}^{\infty} a_n n (x - a)^{n-1}. \text{ Similarly, we have } f''(x) = \sum_{n=1}^{\infty} a_n n(n-1)(x - a)^{n-2} \text{ and}$$

$$f'''(x) = \sum_{n=1}^{\infty} a_n n(n-1)(n-2)(x - a)^{n-3}.$$

If we substitute  $x = a$  into each of the above series all the terms will cancel except one and we get

$$f(a) = a_0$$

$$f'(a) = a_1$$

$$f''(a) = 2a_2$$

$$f'''(a) = 3!a_3$$

In general, we have

$$f^{(n)}(a) = n!a_n.$$

Solving for  $a_n$ , we have

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

If we substitute this back into our original series for  $f(x)$ , we get the **Taylor series of  $f(x)$  about the point  $x = a$** :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

If  $x = 0$ , then the series is called the **Maclaurin series of  $f$** . This series has the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

**Example 1: Finding Taylor Series**

Find a Taylor series for the function  $f(x) = e^x$  about the point  $x = 0$ .

**Example 2: Finding Taylor Series**

Find a Taylor series for the function  $f(x) = \sin x$  about the point  $x = 0$ .

**Example 3: Finding Taylor Series**

Find a Taylor series for the function  $f(x) = \cos x$  about the point  $x = 0$ .

**Definition: Taylor Polynomial**

If we terminate the Taylor series for a given function after  $n$  terms, we get the  $n$ th degree Taylor polynomial for  $f$ .

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} (x-a)^n$$

This polynomial can be used to approximate  $f(x)$ . How much is the error in any approximation? The next theorem provides the answer to this question.

**Theorem: Taylor's Theorem** Suppose that  $f(x)$  has  $n+1$  derivatives on the interval  $|x-a| < r$  for some  $r > 0$ . Then, for  $x \in (a-r, a+r)$ ,  $P_n(x) \approx f(x)$  and the error in using  $P_n(x)$  to approximate  $f(x)$  is

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} (x-a)^{n+1},$$

for some  $z$  between  $x$  and  $a$ .

**Theorem** Suppose that  $f(x)$  has derivatives of all orders on the interval  $|x - a| < r$  for some  $r > 0$  and  $\lim_{n \rightarrow \infty} R_n(x) = 0$ , for all  $x \in (a - r, a + r)$ , Then the Taylor series for  $f$  expanded about  $x = a$  converges to  $f(x)$ , that is,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n ,$$

for all  $x \in (a - r, a + r)$ .

**Theorem: Taylor's Inequality**

If  $|f^{(n+1)}(x)| \leq M$  for  $|x - a| \leq d$ , then the remainder  $R_n(x)$  of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n + 1)!} |x - a|^{n+1} \text{ for } |x - a| \leq d$$

**Example 4: Finding Taylor Series**

Find a Taylor series for the function  $f(x) = \ln x$  about the point  $x = 1$  and its interval of convergence. Use the 4<sup>th</sup> degree Taylor polynomial,  $P_4(x)$ , to approximate the value of  $\ln(1.1)$  and find the error in this approximation.

**Example 5: Finding a Taylor Series**

Find a Taylor series for the function  $f(x) = \frac{1}{(1+x)^2}$  about the point  $x = 0$  and its interval of convergence.

**Example 6: Finding Taylor Series from Known Series**

Find Taylor series for the following functions  $f(x) = xe^{-x^2}$  about  $x = 0$  using known Taylor series and find the interval of convergence.

**Example 7: Finding Taylor Series from Known Series**

Find Taylor series for the following functions  $f(x) = \sin x^2$  about  $x = 0$  using known Taylor series and find the interval of convergence.

**Example 8: Finding a Taylor Polynomial**

Find the Taylor polynomial of degree 4 at  $x = 0$  for the function defined by  $f(x) = \ln(1 + x)$ . Then compute the value of  $\ln(1.1)$  to as many decimal places as the polynomial of degree 4 allows.