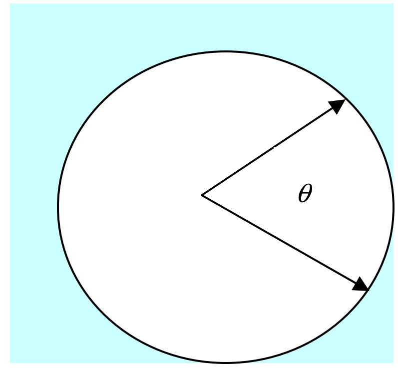
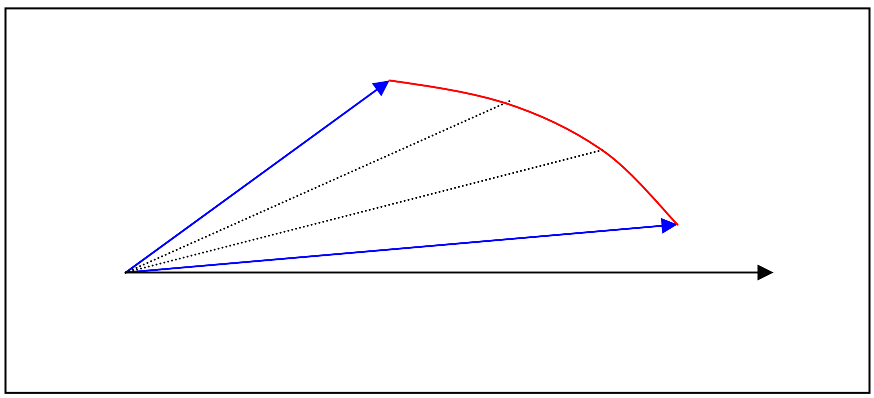
Section 3 Areas and Lengths in Polar Coordinates

Area of a Polar Region

Recall that the area of a sector of a circle is $\frac{1}{2}r^2\theta$.



The area of a small region bounded by a polar curve can be approximated by the area formula above:



We begin by dividing the angle from $a = \theta_0$ to $b = \theta_n$ into n equal size angles of width $\Delta\theta$. Over the small interval from $\theta = \theta_{i-1}$ to $\theta = \theta_i$ we can think of r as having a constant value $r_i = f(\theta_i^*)$ where θ_i^* can be any point in the interval since we will be dividing the angle into more and more subintervals. Then the area of the small sector is given by $\Delta A = \frac{1}{2}r_i^2\Delta\theta$.

By adding the areas of all these sectors we get an approximation to the area of the entire region.

$A \approx \sum_{i=0}^{n}\frac{1}{2}r_i^2\Delta\theta$. The exact area can be found by taking the limit as $n \to \infty$.

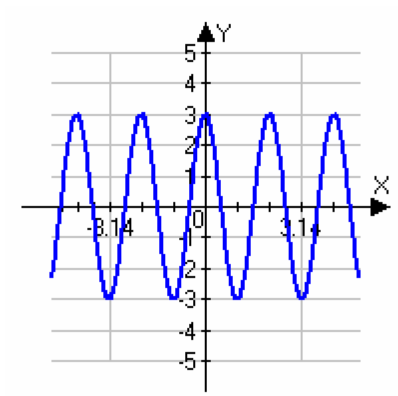
> **Theorem: Area of a Polar Region**. If $r = f(\theta)$ then the area A of a polar region R bounded by $r = f(\theta)$, $\theta = a$, and $\theta = b$ is given by
>
> $$A = \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta = \int_a^b \frac{1}{2} r^2 d\theta$$

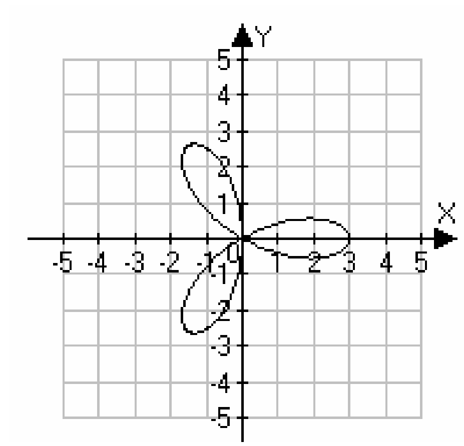
Example 1: Finding the Area of a Polar Region

Find the area of one petal of the rose curve given by $r = 3\cos 3\theta$.

Solution:

We begin by drawing the graph:





We see that one petal is traced out as θ goes from $a = -\frac{\pi}{6}$ to $b = \frac{\pi}{6}$.

So the area is given by

$$A = \int_a^b \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_{-\pi/6}^{\pi/6} (3\cos 3\theta)^2 \, d\theta = \frac{9}{2} \int_{-\pi/6}^{\pi/6} \cos^2 3\theta d\theta$$

$$= \frac{9}{2} \int_{-\pi/6}^{\pi/6} \frac{(1 + \cos 6\theta)}{2} d\theta = \frac{9}{4} \int_{-\pi/6}^{\pi/6} (1 + \cos 6\theta) \, d\theta$$

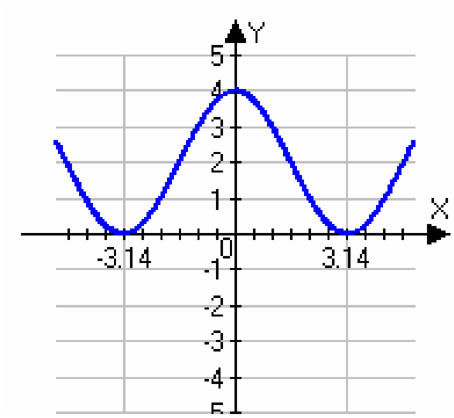
$$= \frac{9}{4}\left[\theta + \frac{\sin 6\theta}{6}\right]_{-\pi/6}^{\pi/6} = \frac{9}{4}\left[\frac{\pi}{6} + \frac{\sin \pi}{6} - \left(-\frac{\pi}{6} + \frac{\sin(-\pi)}{6}\right)\right] = \frac{3\pi}{4}$$

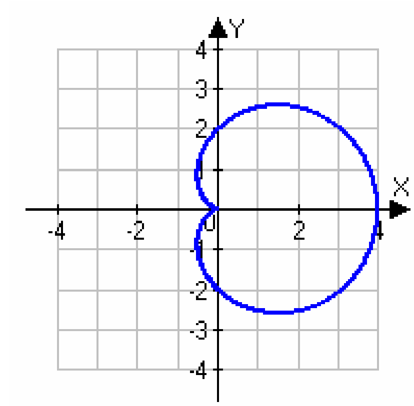
Example 2: Finding the Area of a Polar Region

Find the area of the region enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Solution:

Graph the function in rectangular coordinates and then graph in polar coordinates.





We see at once that the area is swept out once as θ goes from 0 to 2π. The area is therefore:

$$A = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} 4(1+\cos\theta)^2 \, d\theta$$

$$= \int_0^{2\pi} 2\left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= \int_0^{2\pi} \left(2 + 4\cos\theta + 2\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta)\, d\theta$$

$$= 3\theta + 4\sin\theta + \frac{1}{2}\sin 2\theta \Big]_0^{2\pi}$$

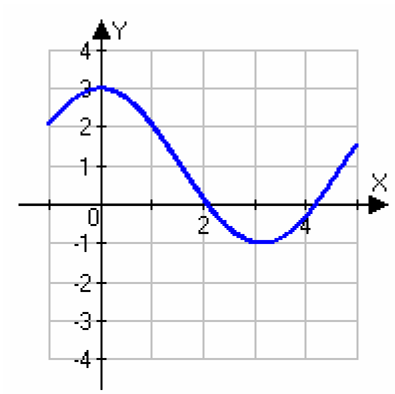
$$= 3\cdot 2\pi + 4\sin(2\pi) + \frac{1}{2}\sin(2\cdot 2\pi) - \left(3\cdot 0 + 4\sin 0 + \frac{1}{2}\sin(2\cdot 0)\right)$$

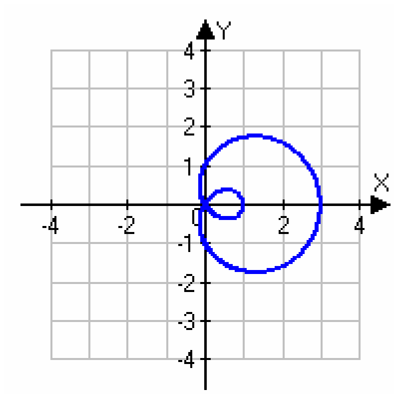
$$= 6\pi$$

Example 3: Finding the Area of a Polar Region

Find the area inside the smaller loop of the limaçon $r = 2\cos\theta + 1$.

Solution: Graph the function in rectangular coordinates and then graph in polar coordinates.





The smaller loop is traced out as θ goes from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$.

The area is therefore:

$$A = \int_{2\pi/3}^{4\pi/3} \frac{1}{2} r^2 d\theta = 2\int_{2\pi/3}^{\pi} \frac{1}{2} r^2 d\theta$$

$$= \int_{2\pi/3}^{\pi} r^2 d\theta = \int_{2\pi/3}^{\pi} (2\cos\theta + 1)^2 \, d\theta$$

$$= \int_{2\pi/3}^{\pi} \left(4\cos^2\theta + 4\cos\theta + 1\right) d\theta$$

$$= \int_{2\pi/3}^{\pi} \left(4\frac{1+\cos 2\theta}{2} + 4\cos\theta + 1\right) d\theta$$

$$= \int_{2\pi/3}^{\pi} (3 + 2\cos 2\theta + 4\cos\theta) \, d\theta$$

$$= 3\theta + \sin 2\theta + 4\sin\theta \Big|_{2\pi/3}^{\pi}$$

$$= 3\pi + \sin 2\pi + 4\sin\pi - 3\frac{2\pi}{3} + \sin\frac{4\pi}{3} + 4\sin\frac{2\pi}{3}$$

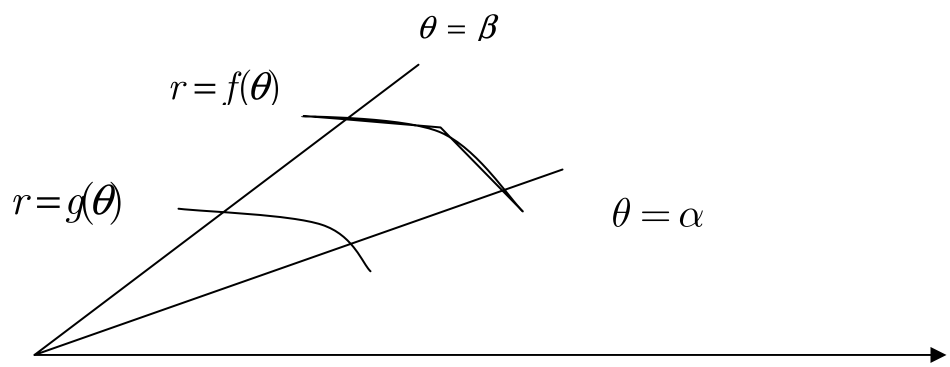
$$= \pi - \frac{\sqrt{3}}{2} + 4\frac{\sqrt{3}}{2}$$

$$= \pi - \frac{3\sqrt{3}}{2}$$

Finding the Area of a Region between Two Curves

Assume that $r = f(\theta)$ and $r = g(\theta)$ describe two curves in polar coordinates and that $f(\theta) \geq g(\theta)$ for $\theta \in [\alpha, \beta]$. Let $\Re$ be the region between these two curves and the rays $\theta = \alpha$ and $\theta = \beta$. The area of $\Re$ is obtained by subtracting the area within the inner curve $r = g(\theta)$ from the area within the outer curve $r = f(\theta)$. That is, the area between two curves is

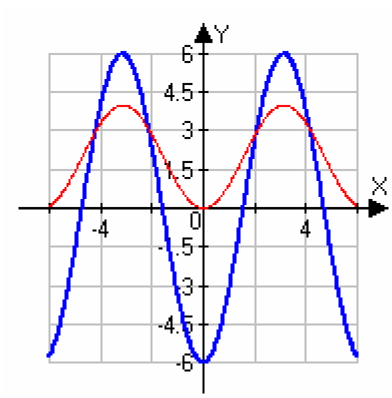
$$A = \int_\alpha^\beta \tfrac{1}{2}[f(\theta)]^2\, d\theta - \int_\alpha^\beta \tfrac{1}{2}[g(\theta)]^2\, d\theta = \tfrac{1}{2}\int_\alpha^\beta \left\{[f(\theta)]^2 - [g(\theta)]^2\right\} d\theta$$

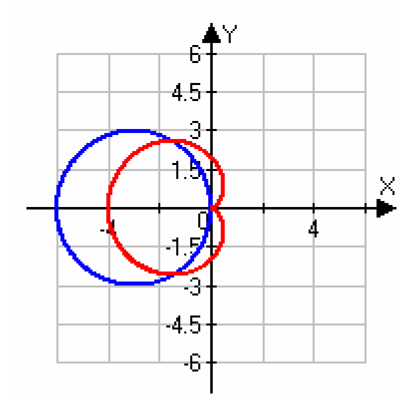


Example 4 Finding the Area of a Region between Two Curves

Find the area of the region inside the circle $r = -6\cos\theta$ and outside the cardioid $r = 2 - 2\cos\theta$.

Solution: We first graph the two functions: in rectangular coordinates and in polar coordinates.





The two graphs intersect when $\theta = \frac{2\pi}{3}$ and when $\theta = \frac{4\pi}{3}$.

Therefore, the area is

$$A = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}\left(r_1^2 - r_2^2\right) d\theta = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}\left[(-6\cos\theta)^2 - (2 - 2\cos\theta)^2\right] d\theta$$

$$= \frac{1}{2}\int_{2\pi/3}^{4\pi/3} \left[36\cos^2\theta - 4 + 8\cos\theta - 4\cos^2\theta\right] d\theta = \int_{2\pi/3}^{4\pi/3} \left[16\cos^2\theta - 2 + 4\cos\theta\right] d\theta$$

$$= \int_{2\pi/3}^{4\pi/3} \left[8(1 + \cos 2\theta) - 2 + 4\cos\theta\right] d\theta = 2\int_{2\pi/3}^{4\pi/3} \left[4(1 + \cos 2\theta) - 1 + 2\cos\theta\right] d\theta$$

$$= 2\left\{\left[4(\theta + \frac{1}{2}\sin 2\theta)\right] - \theta + 2\sin\theta\right\}_{2\pi/3}^{4\pi/3}$$

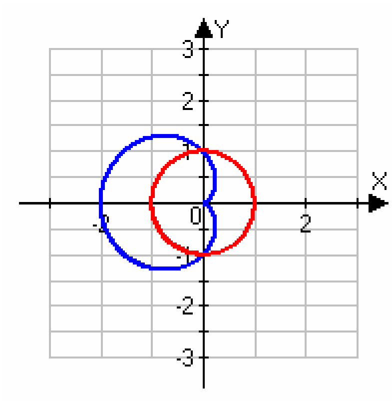
$$= 2\left\{\left[\frac{16\pi}{3} + 2\sin\frac{8\pi}{3})\right] - \frac{4\pi}{3} + 2\sin\frac{4\pi}{3}\right\} - 2\left\{\left[(\frac{8\pi}{3} + 2\sin\frac{4\pi}{3})\right] - \frac{2\pi}{3} + 2\sin\frac{2\pi}{3}\right\}$$

$$= \frac{32\pi}{3} + 4\sin\frac{8\pi}{3} - \frac{8\pi}{3} + 4\sin\frac{4\pi}{3} - \frac{16\pi}{3} - 4\sin\frac{4\pi}{3} + \frac{4\pi}{3} - 4\sin\frac{2\pi}{3}$$

$$= 4\pi$$

Example 5 Finding the Area of a Region between Two Curves

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.



$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_1^2 - r_2^2\right) d\theta$$
$$= 2\int_0^{\pi/2} \frac{1}{2}\left(r_1^2 - r_2^2\right) d\theta$$
$$= \int_0^{\pi/2} \left(2\cos\theta - \cos^2\theta\right) d\theta$$
$$= 2 - \frac{\pi}{4}$$

Finding the Arc Length of a Polar Curve

Theorem: Arc Length of a Polar Curve. Let f be a function whose derivative is continuous in an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = a$ to $\theta = b$ is given by

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Proof: We derive the formula above by using the arc length formula for a parametric curve.

Let a point be given by $(r, \theta) = (f(\theta), \theta)$.

The parametric equations of the point are $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$.

The curve is now given in rectangular form with parameter θ. Its length is

$$s = \int_{\alpha}^{\beta} \sqrt{\left[\frac{dx}{d\theta}\right]^2 + \left[\frac{dy}{d\theta}\right]^2}\, d\theta.$$

Now, $\frac{dx}{d\theta} = f(\theta)(-\sin\theta) + f'(\theta)\cos\theta$ and $\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta$, hence

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2\sin^2\theta - 2f(\theta)f'(\theta)\sin\theta\cos\theta + [f'(\theta)]^2\cos^2\theta +$$

$$[f(\theta)]^2\cos^2\theta + 2f(\theta)f'(\theta)\sin\theta\cos\theta + [f'(\theta)]^2\sin^2\theta,$$

which simplifies to $[f(\theta)]^2 + [f'(\theta)]^2$.

Example 6 Finding the Arc Length of a Polar Curve

Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the cardioid $r = f(\theta) = 2 - 2\cos\theta$.

Solution:

Since $f'(\theta) = 2\sin\theta$, we have

$$s = \int_0^{2\pi} \sqrt{(2 - 2\cos\theta)^2 + (2\sin\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4 - 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{8 - 8\cos\theta}\, d\theta = \sqrt{8}\int_0^{2\pi} \sqrt{1 - \cos\theta}\, d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

$$= \sqrt{8}\int_0^{2\pi} \sqrt{2\sin^2\frac{\theta}{2}}\, d\theta$$

$$= 4\int_0^{2\pi} \sqrt{\sin^2\frac{\theta}{2}}\, d\theta$$

$$= 4\int_0^{2\pi} \left|\sin\frac{\theta}{2}\right| d\theta$$

$$= -8\cos\left(\frac{\theta}{2}\right)\Bigg|_0^{2\pi} = -8\cos(\pi) + 8\cos 0 = 16$$

Example 7 Finding the Arc Length of a Polar Curve

Find the length of the cardioid $r = 1 - \cos\theta$.

Recall that arclength is $L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

Solution:

Here $r = 1 - \cos\theta$ and $\frac{dr}{d\theta} = \sin\theta$,

so that

$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 2 - 2\cos\theta$ and

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$$

$$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta = -4\cos\frac{\theta}{2}\Big]_0^{2\pi} = 8$$