Name Key

Project 2
Calculus 2
MATH 2414
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Directions: Please do the work on scrap and then copy all finished work ***with detailed explanations*** neatly onto paper.
You may share ideas but you may not copy another student's work. You may not get help from the Learning Lab on these questions.

Deadline: ______________________

No help from outside sources. Use only your classmates, book, and professor!

(5 points)

1. Answer the following questions about sequences.

a) What is the formula for the general term $\{a_n\}$ of the sequence $\left\{1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \cdots\right\}$? $a_n = \frac{(-1)^n}{2^n}$

b) Determine the limit of the sequence $a_n = \sqrt{n+1} - \sqrt{n}$.

$a_n = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0$

c) Determine the limit of the sequence $a_n = (-1)^n \left(1 - \frac{1}{\sqrt{n}}\right)$ .if it exists. If it does not state that the sequence diverges.

diverges by oscillation

d) Determine the limit of the sequence $a_n = \ln(2n+1) - \ln n$.

$a_n = \ln\left(\frac{2n+1}{n}\right) = \ln\left(2 + \frac{1}{n}\right) \to \ln 2$

e) Assume that the limit of the sequence $\left\{\sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, \cdots\right\}$ exists, then what must be the limit?

3

(6 points)

2. Give an example of each of the following:

   a) A sequence which is unbounded. $a_n = n$

   b) A sequence which is bounded, but does not converge. $a_n = (-1)^n$

   c) A sequence which converges, but for which the terms are those of a divergent series. $a_n = \frac{1}{n} \to 0$ but $\sum \frac{1}{n}$ diverges

(6 points)

3. Determine whether each sequence converges or diverges, giving a reason, and if it does converge, find its limit.

   a) $a_n = \dfrac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})(\sqrt{n+1} + \sqrt{n})} = \dfrac{1}{(\sqrt{n+1} + \sqrt{n})^2} \to 0$ converges

   b) $a_n = \left(\dfrac{n-5}{n}\right)^n$ Let $y = \left(\frac{x-5}{x}\right)^x = \left(1 - \frac{5}{x}\right)^x$

   then $\ln y = \dfrac{\ln(1 - \frac{5}{x})}{\frac{1}{x}}$. As $x \to \infty$ both parts of fraction go to 0 so L'H's rule applies

   $$\lim_{x \to \infty} \frac{\ln(1 - \frac{5}{x})}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\frac{1}{1 - \frac{5}{x}}\left(\frac{5}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x \to \infty} \frac{-5}{1 - \frac{5}{x}} = -5$$

   since $\ln y \to -5$ $y \to e^{-5}$

   $\therefore \lim_{n \to \infty} a_n = e^{-5}$

(15 points)

4. Find the sum of each of the following convergent series.

a) $\frac{99}{88} - \frac{9}{11} + \frac{72}{121} - \cdots = \frac{99/88}{1+\frac{8}{11}} = \frac{99}{152}$

geometric w/ $|r| = \left|-\frac{8}{11}\right| < 1$

b) $1 + \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \cdots = \frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}$

$a = 1$

$|r| = |1/e| < 1$ convergent geometric

c) $\sum_{n=1}^{\infty} \left(\frac{4}{2^n} - \frac{2}{n(n+1)} \right) = 2$

d) $\sum_{n=1}^{\infty} e^{-2n} = \frac{1}{e^2} + \frac{1}{e^4} + \frac{1}{e^6} + \cdots = \frac{1/e^2}{1 - 1/e^2} = \frac{1}{e^2-1}$

$a = 1/e^2$

$0 < r = 1/e^2 < 1$

e) $\sum_{n=0}^{\infty} \frac{4}{(4n-3)(4n+1)} = -\frac{1}{3}$

(9 points)

5. Determine whether each series is convergent or divergent. Give reasons for your answers.

a. $\sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right)$ diverges by n^{th} term test since $a_n = \ln\left(\frac{1}{n}\right) \to -\infty$

b. $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n = \frac{e/\pi}{1-\frac{e}{\pi}} = \frac{e}{\pi - e}$

geometric w $0 < r = \frac{e}{\pi} < 1$

c. $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$ Diverges by n^{th} term test, since

$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1$

Let $x = \frac{1}{n}$

(9 points)

6. Determine whether each series is convergent or divergent. **Carefully explain why for full credit.**

a) $$\sum_{n=1}^{\infty} \frac{1}{n^2 + \ln n}$$

$\ln n > 0$ if $n > 1$
so that if $n > 1$

$n^2 + \ln n > n^2$

$$\frac{1}{n^2 + \ln n} < \frac{1}{n^2}$$

and the series converges by direct comparison w/ the convergent p-series $\sum \frac{1}{n^2}$ ($p=2$)

b) $$\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - \frac{1}{3^n}$$ difference of two convergent geometric series.

$$= \frac{1}{1 - \frac{2}{3}} - \frac{1}{1 - \frac{1}{3}} = 3 - \frac{3}{2} = 3/2$$

c) $$\sum_{n=1}^{\infty} \frac{5^n}{(2n+1)!}$$

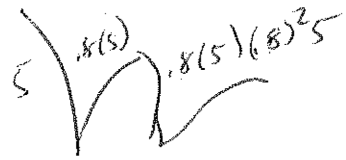
Ratio Test

$$\left| \frac{5^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{5^n} \right| = \left| \frac{5}{(2n+3)(2n+2)} \right| \xrightarrow[n\to\infty]{} 0$$ converges by the ratio test.

No help from outside sources. Use only your classmates, book, and professor!

(8 points)

7. A ball is dropped from a height of 5 m. Each time it hits the sidewalk from a height of h meters it bounces back up to a height of $0.8h$ meters. Find the total distance traveled by the ball going up *and* down.



$$5 + 2\sum_{n=1}^{\infty}(0.8)^n 5$$

$$5 + 2\frac{4}{1-0.8} = 5 + 2(20) = 45 \text{ meters}$$

(7 points)

8. Approximate the sum of the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ by the 30th partial sum and estimate the error in your approximation.

$$S_{30} = 0.947032253252$$

$$|\text{Error}| < a_{31} = \frac{1}{31^4} = 0.000001$$

(4.5 points)

9. Consider the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{4^n}$.

$|a_n| = \frac{n}{4^n}$

(a) Show that the series is absolutely convergent.

Ratio Test $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+1}{4^{n+1}} \cdot \frac{4^n}{n}\right| = \left|\frac{1}{4}\frac{n+1}{n}\right| \xrightarrow[n\to\infty]{} \frac{1}{4} < 1$

(b) Calculate the sum of the first 32 terms to approximate the sum of the series.

$S_{32} = 0.2$

(c) Is the approximation in part (b) an overestimate or an underestimate?

underestimate

(d) Estimate the error involved in the approximation from part (b).

$|\text{Error}| < a_{33} = \frac{33}{4^{33}} \approx 4.47 \times 10^{-19}$

(12 points)

10. Determine whether each series converges absolutely, converges conditionally, or diverges. Carefully explain your reasoning for full credit.

a) $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

$\left|\frac{\cos n}{n^2}\right| < \frac{1}{n^2}$

Converges absolutely by the comparison test with the p-series $\sum \frac{1}{n^2}$

c) $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n!}$

Ratio Test $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}\right|$

$= \left|\frac{3}{n+1}\right| \to 0$

Converges absolutely.

b) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$

$-1, 1, -1, 1$

$= \sum \frac{(-1)^n}{n}$

Converges conditionally by AST while $\sum \frac{1}{n}$ is the divergent harmonic series.

d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

Converges conditionally

The series converges by the AST, while $\sum \frac{1}{\sqrt{n+1}}$ diverges by LCT with $\sum \frac{1}{\sqrt{n}}$ a divergent p series ($p = 1/2 < 1$)

(7.5 points)

11. This is a set of true/false questions. To answer, please place T or F before the letter of each problem. Choose carefully. Hint: One way to show a statement is false is by providing a counterexample.

F a) The series $\sum_{n=1}^{\infty} a_n$ converges whenever $\lim_{n\to\infty} a_n = 0$.

T b) The series $\sum_{n=1}^{\infty} a_n$ converges whenever the sequence of partial sums converges to 0.

T c) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, then $\sum_{n=1}^{\infty} a_n^2$ converges.

F d) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, then $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges.

F e) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, then $\sum_{n=1}^{\infty} \frac{1+a_n}{2+a_n}$ converges.

T f) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, then $\sum_{n=1}^{\infty} \frac{2^n + a_n}{3^n + a_n}$ converges.

T g) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ converges.

F h) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$ converges.

T i) If p is a fixed real number, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

F j) The series $\sum_{n=1}^{\infty} n^{\cos 3}$ converges.

F k) $\sum_{n=1}^{\infty} \frac{n+1}{n+3}$ converges to 1/3.

T l) $\sum_{n=2}^{\infty} 3(4)^{-n+1}$ converges to 1.

T m) If $a_n > 0$ and the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ is bounded above, then the series converges.

F n) If the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ is bounded, then the series converges.

F o) If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} |a_n| = B$ and A and B are finite, then $|A| = B$.

(5 points)

12. A cable 100 meters long and weighing 3 Newtons/meter hangs over a cliff. Find the work done in raising the cable to the top of the cliff.

15,000 Joules

(6 points)

13. A tank containing 1000 lb of water is being lifted to the top of a building 80 ft high by a cable 80 ft long and weighing 160 lb. How much work is done in raising the tank of water 50 ft if water is leaking out of the tank at a rate of one pound for each foot the tank is raised?

54,250 ft/lbs