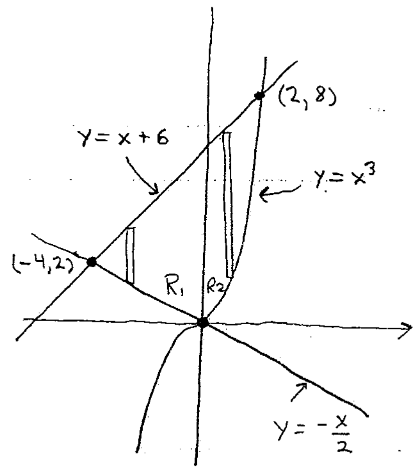
Example: Find the area of the region R that is bounded by the graphs of $y - x = 6$, $y - x^3 = 0$, and $2y + x = 0$.



Note: Here we cannot find the area using only one definite integral. We must integrate over the two subregions R_1 and R_2.

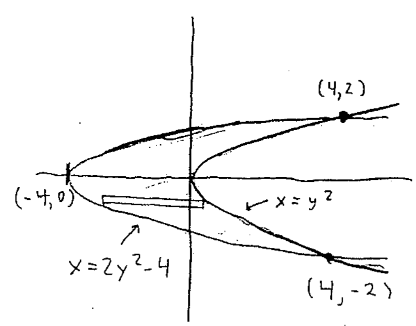
$A = A_1 + A_2$

$$A_1 = \int_{-4}^{0} \left[(x+6) - \left(-\frac{x}{2}\right)\right] dx = \int_{-4}^{0} \left[\frac{3}{2}x + 6\right] dx = \frac{3}{4}x^2 + 6x \Big|_{-4}^{0} = 0 - (12 - 24) =$$

$$A_2 = \int_{0}^{2} \left[x + 6 - x^3\right] dx = \left(\frac{x^2}{2} + 6x - \frac{x^4}{4}\right)\Big|_{0}^{2} = (2 + 12 - 4) - 0 = 10.$$

So the area of the entire region is $A = A_1 + A_2 = 12 + 10 = 2$

Example: Find the area of the region bounded by the graph of the equations $2y^2 = x + 4$ and $x = y^2$,



Let $f(y) = y^2$ $\quad g(y) = 2y^2 - 4$

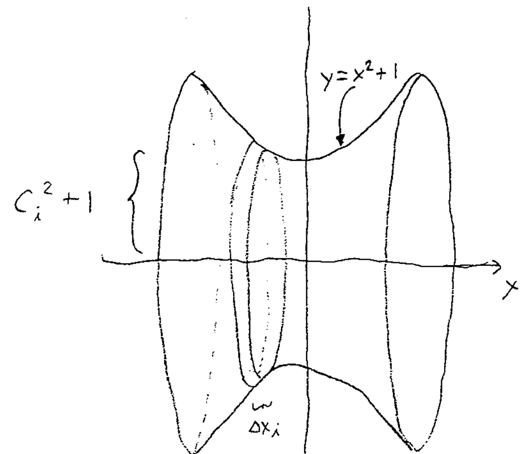
$$A = \int_{-2}^{2} \left[y^2 - (2y^2 - 4)\right] dy$$

$$= \int_{-2}^{2} \left[4 - y^2\right] dy = \left(4y - \frac{y^3}{3}\right)\Big|_{-2}^{2}$$

$$= \left[8 - \frac{8}{3}\right] - \left[-8 - \left(-\frac{8}{3}\right)\right] = \frac{32}{3}.$$

Example:

If $f(x) = x^2+1$, find the volume of the solid generated by revolving the region under the graph of f from -1 to 1 about the x-axis.



volume of each disk is $\pi(c_i^2+1)^2$

$$V = \pi\int_a^b [R(x)]^2\,dx$$

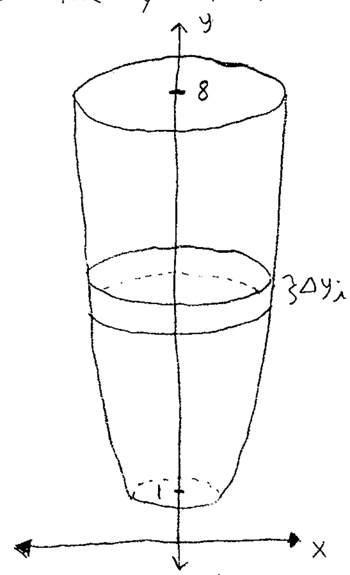
$$= \pi\int_{-1}^{1} (x^2+1)^2\,dx$$

$$= \pi\int_{-1}^{1} (x^4+2x^2+1)\,dx$$

$$= \pi\left[\frac{x^5}{5} + \frac{2}{3}x^3 + x\right]_{-1}^{1}$$

$$= \pi\left[\left(\frac{1}{5}+\frac{2}{3}+1\right) - \left(-\frac{1}{5}-\frac{2}{3}-1\right)\right] = \frac{56}{15}\pi$$

Example: The region bounded by the y-axis and the graphs of $y = x^3$, $y=1$, and $y=8$ is revolved about the y-axis. Find the volume of the resulting solid.



radius of typical
is $x = y^{1/3}$

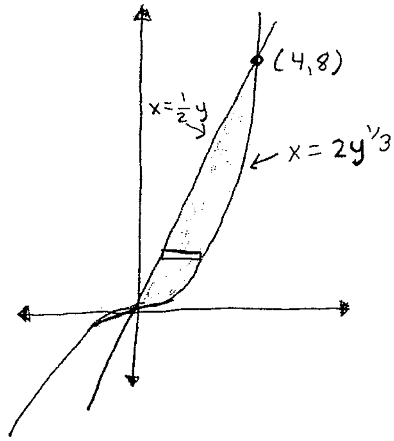
$$V = \int_1^8 \pi(y^{1/3})^2\,dy = \pi\int_1^8 y^{2/3}\,dy$$

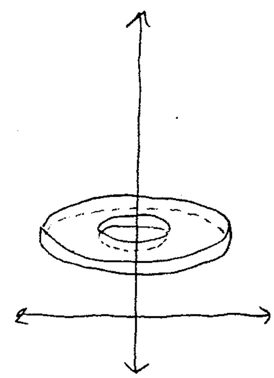
$$= \pi\left(\frac{3}{5}\right)\left[y^{5/3}\right]_1^8$$

$$= \frac{3}{5}\pi\left[8^{5/3} - 1\right]$$

$$= \frac{93}{5}\pi.$$

Example: The region in the first quadrant bounded by the graphs of $y = \frac{1}{8}x^3$ and $y = 2x$ is revolved about the y-axis. Find the volume of the resulting solid.





outer radius = $2y^{1/3}$

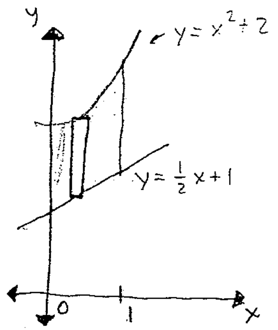
inner radius = $\frac{1}{2}y$

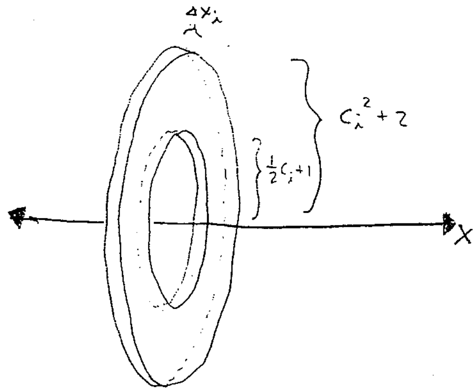
$$V = \int_0^8 \pi\left[(2y^{1/3})^2 - \left(\tfrac{1}{2}y\right)^2\right] dy = \int_0^8 \pi\left[4y^{2/3} - \tfrac{1}{4}y^2\right] dy$$

$$= \pi\left[\tfrac{12}{5}y^{5/3} - \tfrac{1}{12}y^3\right]_0^8 = \pi\left[\tfrac{12}{5}(8^{5/3}) - \tfrac{1}{12}(8^3)\right] = \frac{512}{15}\pi.$$

Example: Using the "washer" method.

The region bounded by the graphs of the equations $x^2 = y-2$, $2y - x - 2 = 0$, $x=0$, and $x=1$ is revolved about the x-axis. Find the volume of the resulting solid.



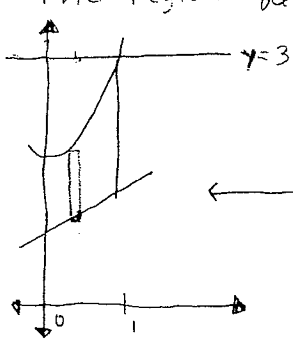


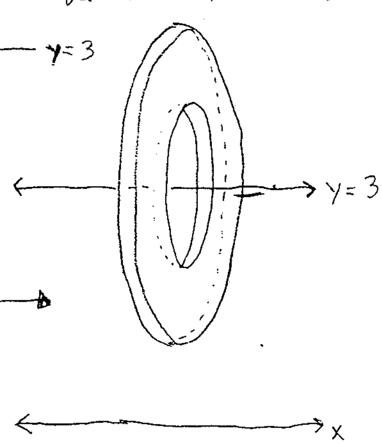
Volume of a washer = $\pi[(\text{outer radius})^2 - (\text{inner radius})^2]\cdot(\text{thickness})$

$$V = \int_0^1 \pi\left[(x^2+2)^2 - \left(\tfrac{1}{2}x+1\right)^2\right]dx$$

$$= \int_0^1 \left(x^4 + \tfrac{15}{4}x^2 - x + 3\right)dx = \pi\left[\tfrac{1}{5}x^5 + \tfrac{5}{4}x^3 - \tfrac{1}{2}x^2 + 3x\right]_0^1 = \frac{79\pi}{20}.$$

Example: Find the volume of the solid generated by revolving the region described above about the line $y=3$.





outer radius $= 3 - \left(\frac{1}{2}x+1\right) = 2 - \frac{1}{2}x$

inner radius $= 3 - (x^2+2) = 1 - x^2$

$$\int_0^1 \pi\left[\left(2-\tfrac{1}{2}x\right)^2 - (1-x^2)^2\right]dx$$

$$= \pi\int_0^1 \left(3 - 2x + \tfrac{9}{4}x^2 - x^4\right)dx$$

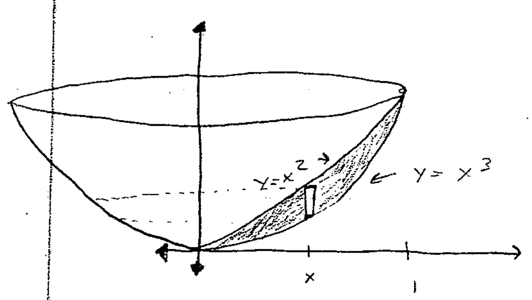
$$= \pi\left[3x - x^2 + \tfrac{3}{4}x^3 - \tfrac{1}{5}x^5\right]_0^1$$

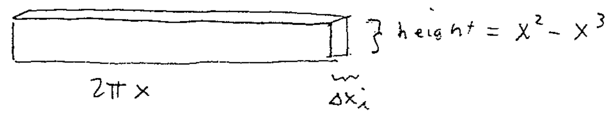
$$= \pi\left[3 - 1 + \tfrac{3}{4} - \tfrac{1}{5}\right] = \frac{51}{20}\pi.$$

(Shells)

Example:

The region between the curves $y = x^2$ and $y = x^3$ is revolved around the y-axis. Find the volume of the solid of revolution produced



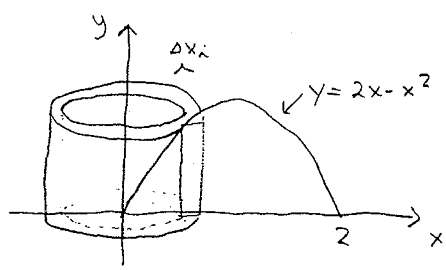


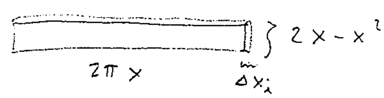
So volume $= \int_0^1 2\pi x(x^2 - x^3)\,dx$

$$= \int_0^1 2\pi(x^3 - x^4)\,dx = 2\pi\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 2\pi\left[\frac{1}{4} - \frac{1}{5}\right]$$

$$= \frac{2\pi}{20} = \frac{\pi}{10}$$

Example: The region bounded by the graph of $y = 2x - x^2$ and the x-axis is revolved about the y-axis. Find the volume of the resulting solid.





Volume $= \int_0^2 2\pi x(2x - x^2)\,dx$

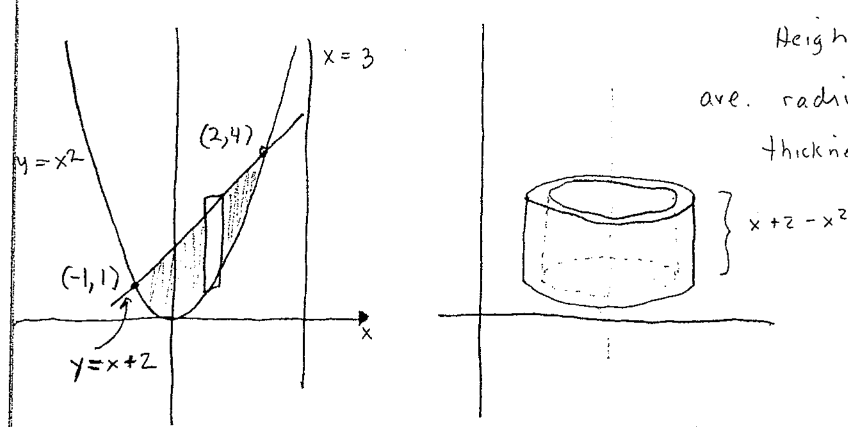
$$= 2\pi \int_0^2 (2x^2 - x^3)\,dx$$

$$= 2\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \frac{8\pi}{3},$$

Could also use washers, but the calculations would be more involved since the given equation would have to be solved for x in terms of y

Example:

The region bounded by the graphs of $y = x^2$ and $y = x+2$ is revolved about the line $x = 3$. Express the volume of the resulting solid as a definite integral.



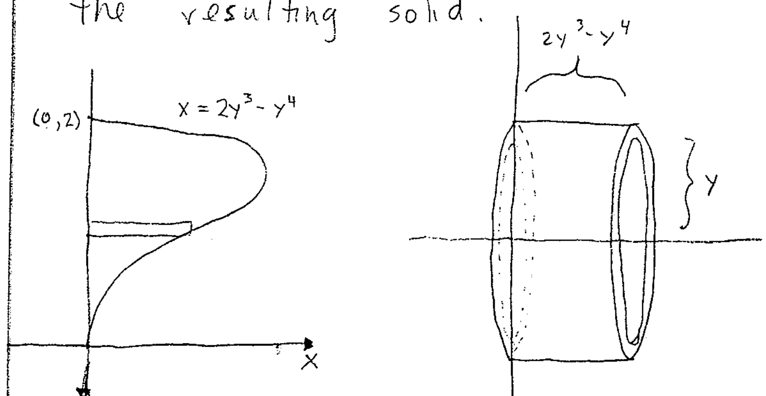
Height $= (x+2) - x^2$

ave. radius $= 3 - x$

thickness $= \Delta x_i$

$$V = \int_{-1}^{2} 2\pi(3-x)(x+2-x^2)\,dx$$

Example: The region in the first quadrant bounded by the graph of $x = 2y^3 - y^4$ and the y-axis is revolved about the x-axis. Set up the integral for the volume of the resulting solid.



[volume of 1 shell is $2\pi y(2y^3 - y^4)$]

$$V = \int_{0}^{2} 2\pi y(2y^3 - y^4)\,dy$$

In this case we are pretty much forced to use shells.

(otherwise we would have to solve $x = 2y^3 - y^4$ explicitly for y in terms

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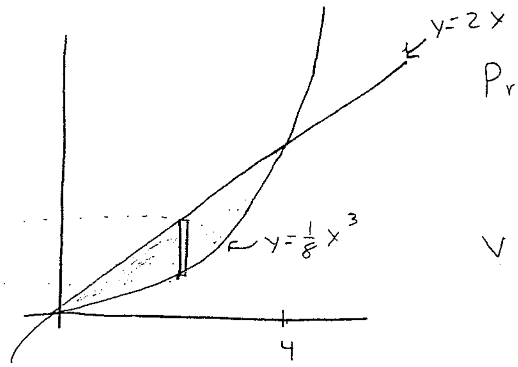

Previously done using washers

$$V = \int_0^4 \left[2\pi x \left(2x - \tfrac{1}{8}x^3\right)\right] dx$$

$$= \pi \int_0^4 \left(4x^2 - \frac{x^4}{4}\right) dx$$

$$= \pi \left[\frac{4}{3}x^3 - \frac{x^5}{20}\right]_0^4 dx$$

$$= \pi \left[\frac{4}{3}(4^3) - \frac{4^5}{20}\right.$$

$$= \pi \left[\frac{256}{3} - \frac{256}{5}\right]$$

$$= 256\pi \left[\frac{5}{15} - \frac{3}{15}\right] = \frac{512\pi}{15}$$

[redacted]