Trig Integral

$$\int_0^{2\pi} \sin^4 x \cos^2 x dx$$

Using the half-angle identities, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$,

we have

$$\begin{aligned}
\int_0^{2\pi} \sin^4 x \cos^2 x dx &= \int_0^{2\pi} \left(\sin^2 x\right)^2 \cos^2 x dx \\
&= \int_0^{2\pi} \left(\frac{1}{2}(1 - \cos 2x)\right)^2 \left(\frac{1}{2}(1 + \cos 2x)\right) dx \\
&= \frac{1}{8}\int_0^{2\pi} \left(1 - 2\cos 2x + \cos^2 2x\right)(1 + \cos 2x)\, dx \\
&= \frac{1}{8}\int_0^{2\pi} \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx
\end{aligned}$$

This integral can be broken into 3 separate integrals:

$$\int_0^{2\pi} (1 - \cos 2x)\, dx = x - \frac{1}{2}\sin 2x \Big|_0^{2\pi} = 2\pi$$

$$\int_0^{2\pi} \cos^2 2x dx = \int_0^{2\pi} \frac{1}{2}(1 + \cos 4x)\, dx = \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right)_0^{2\pi} = \pi$$

$$\int_0^{2\pi} \cos^3 2x dx = \int_0^{2\pi} \cos^2 2x \cos 2x dx = \int_0^{2\pi} \left(1 - \sin^2 2x\right) \cos 2x dx$$

If we let $u = \sin 2x$ then $du = 2cos2x$ and $\frac{1}{2}du = \cos 2x$. Then,

$$\int_0^{2\pi} \left(1 - \sin^2 2x\right) \cos 2x dx = \frac{1}{2}\int_0^{0} \left(1 - u^2\right) du = 0.$$

So, $\int_0^{2\pi} \sin^4 x \cos^2 x dx = \frac{1}{8}\int_0^{2\pi} \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx$

$$= \frac{1}{8}(2\pi - \pi + 0) = \frac{\pi}{8}$$