Section 1 Coordinate Geometry

In this course, we will extend most of the ideas learned in Calculus 1 and 2 to higher dimensions. We can refer to the usual Cartesian plane as $\mathbb{R}^2$. We will develop calculus in three-dimensional space, which we can label $\mathbb{R}^3$, by extending the rectangular coordinate system to 3-space and by introducing alternative coordinate systems that can be used to simplify calculations.

Example 1 Describing Planar Regions

Describe and sketch the regions in the plane $\mathbb{R}^2$ given by the following sets:

a) $\{(x,y) \mid y \geq 1\}$

b) $\{(x,y) \mid |y| > 1\}$

c) $\{(x,y) \mid x^2 + y^2 = 1\}$

d) $\{(x,y) \mid x^2 + y^2 < 1\}$

e) $\{(x,y) \mid x^2 + y^2 > 1\}$

Definition To find the distance between the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, we use the distance formula:

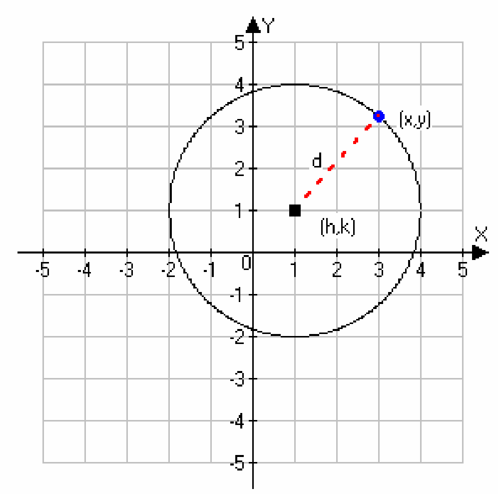
$$d(P_1, P_2) = |P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Definition An **equation of a curve** is an equation that is made true by the coordinates of the points on the curve and by no other points in the plane.

Conic Sections

Circles

A circle is a set of points a fixed distance from a given point called the center. The equation of a circle is found by applying the distance formula using the center of the circle (h,k) and a point on the curve (x,y).



Then the equation is

$d(P_1,P_2) = |P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d = \sqrt{(x-h)^2 + (y-k)^2}$ or

$(x-h)^2 + (y-k)^2 = d^2$. The distance here is precisely the radius of the circle.

Definition A circle is a set of points a fixed distance from a given point. The equation of a circle centered at (h,k) with radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

Example 2 Finding the Center and Radius of a Circle

Find the center and radius of the circle with equation

$$3x^2 + 3y^2 - 12x - 30y + 45 = 0.$$

Solution

$3x^2 + 3y^2 - 12x - 30y + 45 = 0$

$x^2 + y^2 - 4x - 10y + 15 = 0$ Divide by 3

$x^2 + y^2 - 4x - 10y = -15$ Move the constant to the right-hand side by adding -15

$x^2 - 4x + \_\_\_\_ + y^2 - 10y + \_\_\_\_ = -15 + \_\_\_\_ + \_\_\_\_$ Pair terms containing x and y

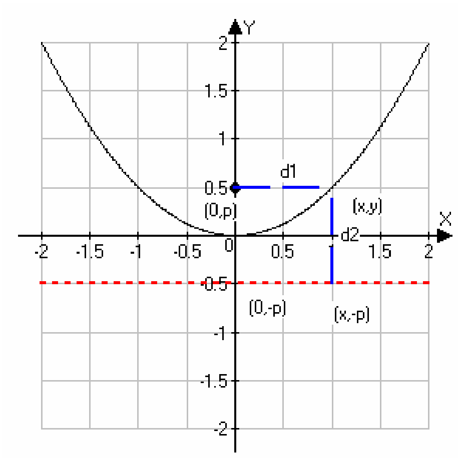
$x^2 - 4x + 4 + y^2 - 10y + 25 = -15 + 4 + 25$ Add square of $1/2$ the coefficient of x,y terms to both sides.

$(x-2)^2 + (y-5)^2 = 14$

Circle has center $(2,5)$ with radius $r = \sqrt{14}$.

Parabolas

A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**).



$$d_1 = d_2$$
$$\sqrt{(x-0)^2 + (y-p)^2} = \sqrt{[x-(+x)]^2 + [y-(-p)]^2}$$
$$x^2 + (y-p)^2 = 0^2 + (y+p)^2$$
$$x^2 + \cancel{y^2} - 2yp + \cancel{p^2} = \cancel{y^2} + 2yp + \cancel{p^2}$$
$$x^2 - 2yp = 2yp$$
$$x^2 = 4yp$$

Definition: A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**). The equation of a parabola with its vertex at the origin and its directrix parallel to the x – axis is simple to write. If the focus is at $(0, p)$, then the directrix has the equation $y = -p$ and the parabola has the equation $x^2 = 4py$. If we write $a = \frac{1}{4p}$, then the equation of the parabola is

$$y = ax^2$$

The equation $x = ay^2$ represents a sideways parabola. If $a > 0$, then the parabola opens to the right and if $a < 0$, then the parabola opens to the left.

Example 3 Parabolas

a) Show that the graph of $y = \frac{-x^2}{16}$ is a parabola and find its focus and directrix.

b) Find the focus, directrix, and equation of the parabola that passes through the point $(8, 2)$ has vertex $(0, 0)$, and focus on the x-axis.

Ellipses

Definition: An **ellipse** is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant. These two fixed points are called the **foci**. If the foci are on the *x* – axis is simple at $(-c, 0)$ and $(-c, 0)$, and the sum of the distances from a point on the ellipse to the foci is $2a$, then the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $c^2 = a^2 - b^2$. The *x*-intercepts are $\pm a$, the *y*-intercepts are $\pm b$, and the foci are $(\pm c, 0)$.

Example 4 Graphing an Ellipse

a) Find the foci of the ellipse $4x^2 + 9y^2 = 36$ and sketch its graph.

b) Find the foci of the ellipse $x^2 + 4y^2 = 4$ and sketch its graph.

Hyperbolas

Definition: A **hyperbola** is the set of points in a plane the difference of whose distances from two fixed points F_1 and F_2 is a constant. These two fixed points are called the foci. If the foci are on the x-axis at $(-c,0)$ and $(-c,0)$, and the difference of the distances from a point P on the hyperbola to the foci is $|PF_1| - |PF_2| = \pm 2a$, then the equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $c^2 = a^2 + b^2$. A hyperbola will have asymptotes with equations $y = \left(\frac{b}{a}\right)x$ and $y = -\left(\frac{b}{a}\right)x$.

Example 5 Graphing a Hyperbola

a) Find the foci and asymptotes of the hyperbola $x^2 - y^2 = 1$ and sketch its graph.

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b) Graph: $9x^2 - 4y^2 = 36$

If the foci of a hyperbola are on the y-axis at $(0,-c)$ and $(0,c)$, and the difference of the distances from a point P on the hyperbola to the foci is $|PF_1|-|PF_2| = \pm 2a$, then the equation of the hyperbola is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ where $c^2 = a^2 + b^2$. A hyperbola will have asymptotes with equations $y = \left(\frac{a}{b}\right)x$ and $y = -\left(\frac{a}{b}\right)x$.

Example 6 Graphing a Hyperbola

a) Find the foci and asymptotes of the hyperbola $y^2 - x^2 = 1$ and sketch its graph.

b) Find the foci and asymptotes of the hyperbola $18y^2 - 8x^2 - 2 = 0$ and sketch its graph.