$$\sin 2x = 2\sin x\cos x$$
$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$\sin^2 x = \frac{1-\cos 2x}{2}$$
$$\cos^2 x = \frac{1+\cos 2x}{2}$$

Section 3 More Integration Techniques

Trigonometric Integrals

If the integrand contains odd powers of sine or cosine, split off one and use substitution.

Example 1: Trigonometric Integrals with One Odd Power of Sine or Cosine

a) $\int \sin^3 x dx$

$= \int \sin^2 x \sin x dx$

$= \int (1-\cos^2 x)\sin x dx$ Now use the substitution $u = \cos x$.

let $u = \cos x$
$du = -\sin x$

$$= -\int 1-u^2 du = \int u^2 - 1 du = \frac{u^3}{3} - u = \boxed{\frac{\cos^3 x}{3} - \cos x + C}$$

b) $\int_0^{\pi/2} \cos^5 x dx$

$$\cos^4 x = (\cos^2 x)^2 = (1-\sin^2 x)^2 = 1 - 2\sin^2 x + \sin^4 x$$

$$= \int_0^{\pi/2} \cos^4 x \cos x dx$$

$$= \int_0^{\pi/2} (1-2\sin^2 x + \sin^4 x)\cos x dx$$ let $u = \sin x$ $du = \cos x dx$

$$= \int_0^{\pi/2} 1 - 2u^2 + u^4 du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 = \sin x - \frac{2}{3}\sin x + \frac{1}{5}\sin^5 x \Big|_0^{\pi/2}$$

$$= 1 - \frac{2}{3}(1) + \frac{1}{5} = \boxed{\frac{8}{15}}$$

c) $\int \sin^3 x \cos^2 x dx$

$$= \int \sin^2 x \cos^2 x \sin x dx$$

$$= \int (1-\cos^2 x)\cos^2 x \sin x dx$$

Let $u = \cos x$
$du = -\sin x dx$

$$= -\int (1-u^2)u^2 du$$

$$= \int u^4 - u^2 du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \boxed{\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C}$$

If the integrand contains even powers of sine or cosine, use the half-angle identities to simplify the integrand.

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

and

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Example 2: Integrand Contains Even Powers of Sine or Cosine

a) $\int \sin^2 x dx$

$$= \frac{1}{2}\int 1 - \cos 2x dx$$

$$= \frac{1}{2}\left[\int dx - \int \cos 2x dx\right] \Rightarrow -\int \cos 2x dx \quad \text{let } u = 2x, \ \frac{1}{2}du = dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\int \cos(u) du\right]$$

$$\boxed{= \frac{1}{2}x - \frac{1}{4}\sin 2x + c}$$

b) $\int_0^{2\pi} \sin^4 x \cos^2 x dx$

$$= \int_0^{2\pi} \sin^2 x \sin^2 x \cos^2 x dx$$

$$= \int_0^{2\pi} \left(\frac{1 - \cos 2x}{2}\right)^2 \left(\frac{1 + \cos 2x}{2}\right) dx \quad \text{Factor out } \frac{1}{8}. \text{ FOIL.}$$

$$= \frac{1}{8}\int_0^{2\pi} \underbrace{(1 - 2\cos(2x) + \cos^2(2x))}\underbrace{(1 + \cos(2x))} dx \quad \text{FOIL, \& simplify.}$$

$$= \frac{1}{8}\int_0^{2\pi} 1 - 2\cos(2x) + \cos^2(2x) + \cos(2x) - 2\cos^2(2x) + \cos^3(2x) dx$$

$$= \frac{1}{8}\int_0^{2\pi} 1 - \cos(2x) - \cos^2(2x) + \cos^3(2x) dx \quad \text{Break apart.}$$

$$= \frac{1}{8}\int_0^{2\pi} 1 - \cos(2x) dx - \frac{1}{8}\int_0^{2\pi} \cos^2(2x) dx + \frac{1}{8}\int_0^{2\pi} \cos^3(2x) dx$$

① u sub.

② $\int \frac{1 + \cos(4x)}{2} dx$

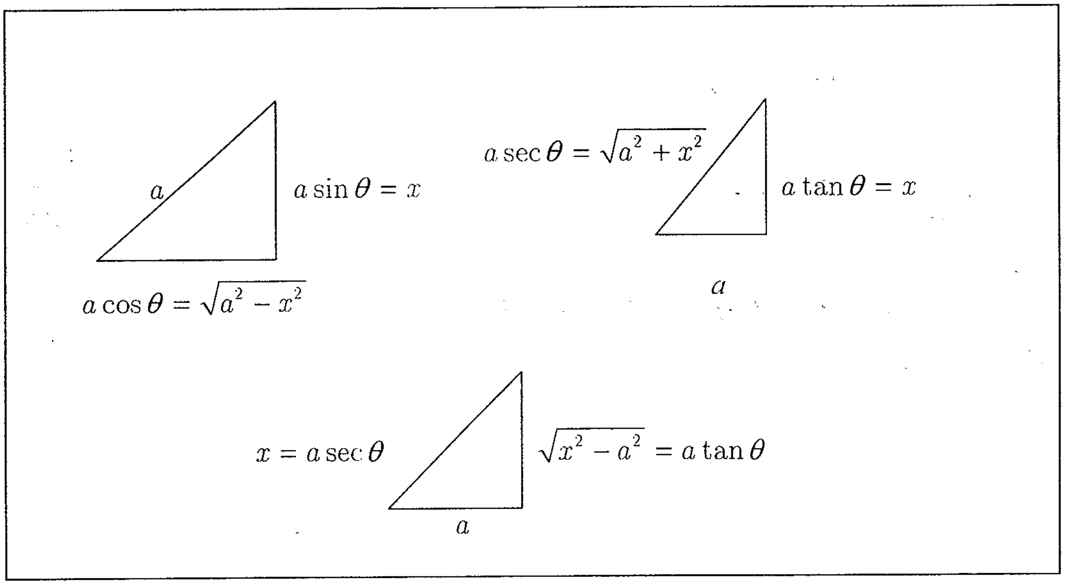
③ $\int \cos^2(2x)\cos(2x) dx$

$= \int (1 - \sin^2(2x))\cos(2x) dx$

Trigonometric Substitutions

It is possible to simplify certain integrals containing radicals by replacing x with a trigonometric function.

If the integrand contains this	Try substituting this
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$



Example 3: Using Trigonometric Substitution

Evaluate $\int \frac{dx}{x^2\sqrt{4-x^2}}$

Hint: Use the substitution $x = 2\sin\theta$

let $\boxed{x = 2\sin\theta}$

$dx = 2\cos\theta\, d\theta$

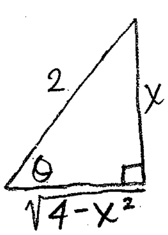
$$= \int \frac{2\cos\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}\, d\theta$$

$$= \frac{1}{2}\int \frac{\cancel{\cos\theta}}{\sin^2\theta \cdot 2\cancel{\cos\theta}}$$

$$= \frac{1}{4}\int \csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + C$$

$$\boxed{-\frac{\sqrt{4-x^2}}{4x} + C}$$

$$\cot\theta = \frac{\sqrt{4-x^2}}{x}$$



$x = 2\sin\theta$

$\sin\theta = \frac{x}{2}$

$x^2 + y^2 = 2^2 \Rightarrow y = \sqrt{4-x^2}$

Example 4: Using Trigonometric Substitution

$\int \frac{\sqrt{x^2-1}}{x} dx$ Hint: Let $x = \sec\theta$.

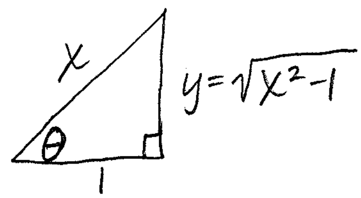
Let $x = \sec\theta$

$dx = \sec\theta \tan\theta \, d\theta$

$\tan = \frac{O}{A} = \frac{\sqrt{x^2-1}}{1}$

$\sec = \frac{H}{A}$

$$= \int \frac{\sqrt{\sec^2\theta - 1}}{\cancel{\sec\theta}} \cancel{\sec\theta} \tan\theta \, d\theta$$



$$= \int \sqrt{\tan^2\theta} \tan\theta \, d\theta$$

$1^2 + y^2 = x^2$

$$= \int \tan^2\theta \, d\theta$$

$y = \sqrt{x^2-1}$

$$= \int \sec^2\theta - 1 \, d\theta = \tan\theta - \theta + C = \boxed{\sqrt{x^2-1} - \arctan\left(\sqrt{x^2-1}\right) + C}$$

Example 5: Using Trigonometric Substitution

$\int \frac{dx}{x^2\sqrt{x^2+4}}$ Hint: Let $x = 2\tan\theta$.

Example 6: Using Trigonometric Substitution

$\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}}$ Hint: Let $x = a\sin\theta$.

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